Conservative Dynamical Systems 2010/2011

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The last exercise is homework, to be handed in on Tuesday 5 October.

4.1 A harmonic n–body problem

The particles A_i with masses m_i $(i = 1, 2, \ldots, n)$ move in three-dimensional space. Any two particles A_i and A_j attract each other by a force of magnitude $F_{ij} = \Gamma m_i m_j d_{ij}$, where $\Gamma > 0$ and d_{ij} denotes the distance $\overline{A_i A_j}$. We suppose that the motions of A_i and A_j are not disturbed if they pass simultaneously through the same point. Determine the general motion of the particles.

4.2 Small oscillations

In a plane Π, a fixed homogeneous rod (length $2a$, mass density s, midpoint O) is given. A particle P of mass M moves in Π. It is attracted by any mass element dm at a point Q on the rod by a force of magnitude $\Gamma M r^\alpha \, dm$. Here, Γ is a positive constant, $\overline{PQ} = r$ and $\alpha = 2n-1$ for $n = 1, 2, 3, \ldots$. Obviously O is a stable equilibrium point. Determine the frequencies of small oscillations of P in the neighbourhood of O.

4.3 Geodesics on a surface of revolution

Let r, φ and z be cylindrical coordinates on $\mathbb{R}^3 = \{x, y, z\}$, so where $x = r\cos\varphi$ and $y = r\sin\varphi$. In the (x, z)–plane a parametrised curve $x = f(v), z = g(v)$ is given, where v varies over an open interval; we assume that here always $f(v) > 0$. Without limitation of generality we also assume that $(f'(v))^2 + (g'(v))^2 = 1$, which expresses that v is an arclength parameter. This curve is revolved around the z–axis, yielding the surface $\mathcal{S}$ parametrised as

$$x \;=\; f(v)\cos\varphi\,, \quad y \;=\; f(v)\sin\varphi\,, \quad z \;=\; g(v)$$

by v and φ. We now investigate when a curve $t \in \mathbb{R} \mapsto \mathbf{R}(t) \in \mathcal{S}$ is a geodesic. By definition the curve $\mathbf{R}$ is a geodesic if for all t

$$\ddot{\mathbf{R}}(t) \perp \mathcal{S}\,.$$

Comment. In the mechanical interpretation we look at a 'free particle' (a point mass of mass 1) moving over $\mathcal{S}$, i.e. without external forces like gravity. According to the d'Alembert principle, the point mass is kept on the surface $\mathcal{S}$ by the perpendicular force $\ddot{\mathbf{R}}(t)$.

1. Show that for a geodesic $t \in \mathbb{R} \mapsto \mathbf{R}(t) \in \mathcal{S}$ one has

$$\begin{aligned} \dot{\mathbf{R}} &= \dot{r}\mathbf{e}_r \;+\; r\dot{\varphi}\mathbf{e}_\varphi \;+\; \dot{z}\mathbf{e}_z \\ \ddot{\mathbf{R}} &= (\ddot{r} - r\dot{\varphi}^2)\mathbf{e}_r \;+\; (2\dot{r}\dot{\varphi} \;+\; r\ddot{\varphi})\mathbf{e}_\varphi \;+\; \ddot{z}\mathbf{e}_z\,. \end{aligned}$$

2. Show that $r^2\dot{\varphi}$ and $\frac{1}{2}\langle \dot{\mathbf{R}} \mid \dot{\mathbf{R}} \rangle = \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2)$ are two (first) integrals of the system and that moreover

$$f'\ddot{r} \;-\; f' r\dot{\varphi}^2 \;+\; g'\ddot{z} \;=\; 0\,.$$

   From now on we write $r(t) = f(v(t))$, $z(t) = g(v(t))$.

–p.t.o.–

3. Show that the statements in item 2 are equivalent to

$$\begin{aligned} 2ff'\dot{v}\dot{\varphi} + f^2\ddot{\varphi} &= 0 \\ \ddot{v} - ff'\dot{\varphi}^2 &= 0\,. \end{aligned}$$

4. Show that from 3, in reverse, it follows that $\ddot{\mathbf{R}}(t) \perp \mathcal{S}$.

5. Define q_1, q_2, p_1 and p_2 by

$$q_1 = v\,,\; q_2 = \varphi\,,\; p_1 = \dot{v}\,,\; p_2 = f^2(v)\dot{\varphi}$$

   and express $H = \frac{1}{2}\langle \dot{\mathbf{R}} \mid \dot{\mathbf{R}} \rangle$ in q_1, q_2, p_1 and p_2. Show that 3 is equivalent to the canonical form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2)\,.$$

   Now re-interprete the conservation laws found under 2.

6. Let $\theta = \theta(t)$ be the angle that the geodesic makes with the 'meridian'. Show that $|f\dot{\varphi}| = |\dot{\mathbf{R}}|\sin\theta$. Next show that $C = f\sin\theta$ is another first integral (this is the celebrated theorem of Clairaut).

7. Show that all meridians of $\mathcal{S}$ are geodesics and that a parallel circle $v = v_0$ of $\mathcal{S}$ is a geodesic precisely when $f'(v_0) = 0$.

8. Fix $p_2 = M$, taking $M \neq 0$. Reduce to one degree of freedom with the effective potential $V_M(q_1) = \frac{M^2}{2f^2(q_1)}$ (compare with the case of the central force field).

   (a) Show that if v_0 is a critical point, then the reduced system has an equilibrium $(q_1, p_1) = (v_0, 0)$. Compare with 7.

   (b) Describe the dynamics of the reduced system near such equilibria in the cases where v_0 is a maximum or a minimum.

   (c) Re-interprete the above findings for the original, unreduced system. Here describe the phase space and its decomposition in invariant level sets $p_2 = M$, $H = E$. What is the geometry of these sets and what is the corresponding dynamics? Also interprete the findings in the configuration space. Why is this description incomplete?

9. Explain the relationship of the items 1 - 5 with the calculus of variations.