Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 3 November.

8.1 Lie derivative and exterior derivative

Prove "Cartan's magic formula"

$$L_X \alpha \;=\; \mathrm{d}(\iota_X \alpha) \;+\; \iota_X \mathrm{d}\alpha$$

where $L_X \alpha$ is the Lie derivative of the differential form α along the vector field X and $\iota_X \alpha$ is the interior product of X with the k–form α, resulting in the $(k-1)$–form

$$\iota_X \alpha(Y_1, \ldots, Y_{k-1}) \;=\; \alpha(X, Y_1, \ldots, Y_{k-1}) \;.$$

8.2 Cotangent lifts

In the Lagrangian setting we can simplify the equations of motion by co-ordinate changes $q \mapsto Q = \varphi(q)$. The corresponding transformations of the velocities and the momenta are determined by φ. In the Hamiltonian setting we can perform more general symplectic co-ordinate changes in phase space $\mathbb{R}^{2n}$. In this exercise we explore the relation between these two types of transformations.

1. Consider a mechanical system in $\mathbb{R}^n$ with co-ordinates $(q_1, \ldots, q_n)$ given by a Lagrangian function $L(q, \dot{q})$. For an arbitrary co-ordinate transformation $q \mapsto Q = \varphi(q)$ compute the corresponding transformation of the momenta $p \mapsto P$. Show that the transformation $(q, p) \mapsto (Q, P) = \Phi(q, p)$ is symplectic with respect to the canonical symplectic form on $\mathbb{R}^{2n}$. The transformation Φ is called the *cotangent lift* of φ.

2. Assume that there is an one-parameter group of diffeomorphisms $\varphi_s : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $L \circ \varphi_s = L$. Show that the corresponding mapping $(q, p) \mapsto \Phi^s(q, p)$ is the flow of a Hamiltonian vector field X_F on $\mathbb{R}^{2n}$ and that $\{F, H\} = 0$ where H is the Legendre transform of L. Show that F is linear in p. What is the relation to Noether's theorem for Lagrangian systems?

8.3 Conditional Liouville measure in the energy level

Given a symplectic manifold $(\mathcal{P}, \omega)$ and a (Hamiltonian) function $H : \mathcal{P} \longrightarrow \mathbb{R}$. Let $c \in \mathbb{R}$ be a regular value of H and consider the level $E_c := H^{-1}(c)$. Show that E_c is a manifold. Of what dimension? Also show that for any $x \in E_c$ the tangent space at E_c is given by $T_x E_c = \ker \mathrm{d}H_x$.

We consider the Liouville volume $2n$–form $\Omega := \omega \wedge \omega \wedge \ldots \wedge \omega$, the n–fold wedge product. It is known that the flow φ_t of the Hamiltonian vector field X_H preserves Ω. Also, the level E_c is preserved by φ_t. The present aim is to construct a 'conditional' volume Ω_c on E_c that is preserved by the restriction $\varphi_t | E_c$. So we consider $x \in E_c$ and tangent vectors $\xi_1, \xi_2, \ldots, \xi_{2n-1} \in T_x E_c$, having to define $\Omega_{c,x}(\xi_1, \xi_2, \ldots, \xi_{2n-1})$. To this end we write the equation

$$\Omega_x(\eta, \xi_1, \xi_2, \ldots, \xi_{2n-1}) \;=\; \mathrm{d}H_x(\eta) \cdot \Omega_{c,x}(\xi_1, \xi_2, \ldots, \xi_{2n-1})$$

where $\eta \in T_x \mathcal{P}$ is arbitrary. Show that this equation determines Ω_c in a unique way, independent of η. Also show that Ω_c is a nondegenerate $(2n-1)$–form, i.e. a volume form on E_c. Finally show that $\varphi_t | E_c$ preserves Ω_c.