Conservative Dynamical Systems 2010/2011

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The last two exercises are homework, to be handed in on Tuesday 30 November.

12.1 The harmonic oscillator

On $\mathbb{R}^2$ consider the Hamiltonian function

$$H(q, p) \;=\; \frac{1}{2}p^2 \;+\; \frac{1}{2}\omega^2 q^2 \;.$$

Give the frequency of the oscillation in the level $H^{-1}(E)$, for $E > 0$. Construct action angle variables (I, ϕ) and determine both $H = H(I)$ and the vector field in terms of (I, ϕ). Then define $\xi := \sqrt{2I}\cos\phi$ and $\eta := \sqrt{2I}\sin\phi$, and rewrite everything in the (ξ, η)–variables. What has changed since the beginning?

12.2 A simple potential

A particle of mass m moves in a plane in a potential given by $V(r) = -V_0$ for $0 \leq r \leq R$ and $V(r) = 0$ for $r > R$, with elastic reflection. Under what initial conditions can the method of action angle variables be applied? Assuming these conditions hold, find the frequencies of the motion.

12.3 The planar isotropic harmonic oscillator

Show that small oscillations of the spherical pendulum near its stable equilibrium position are described by the planar isotropic harmonic oscillator given by the Hamiltonian

$$H(q, p) \;=\; \frac{1}{2}(p_1^2 \;+\; p_2^2 \;+\; q_1^2 \;+\; q_2^2) \;.$$

Compute action angle variables $(I_1, I_2, \phi_1, \phi_2)$ and express the Hamiltonian and the equations of motion in these coordinates. Then express H in terms of polar coordinates (r, θ) in the plane and compute again action angle variables $(I_r, I_\theta, \phi_r, \phi_\theta)$ for H using polar coordinates. Find and interpret the ratio of the radial and angular frequencies of the system.