Conservative Dynamical Systems

The last exercise is homework, to be handed in on 10 March.

5.1 The spherical pendulum

A spherical pendulum has length ℓ and mass m. Let g be the acceleration of gravity.

1. Derive the equations of motion from the variational principle.
2. Determine two (first) integrals, or conserved quantities.
3. Give the Hamiltonian equations for the system, in which the conservation laws are well expressed. Reduce to one degree of freedom (as in the central force field problem).
4. Describe the dynamics of the spherical pendulum in terms of this reduction. First describe the geometry of the invariant level sets defined by the conserved quantities and second characterise the corresponding dynamics. Interprete these findings in the configuration space. Why is this description not complete?

5.2 Geodesics on a surface of revolution

Let r, φ and z be cylindrical coordinates on $\mathbb{R}^3 = \{x, y, z\}$, so where $x = r\cos\varphi$ and $y = r\sin\varphi$. In the (x, z)–plane a parametrised curve $x = f(v), z = g(v)$ is given, where v varies over an open interval; we assume that here always $f(v) > 0$. Without limitation of generality we also assume that $(f'(v))^2 + (g'(v))^2 = 1$, which expresses that v is an arclength parameter. This curve is revolved around the z–axis, yielding the surface $\mathcal{S}$ parametrised as

$$x = f(v)\cos\varphi\,, \quad y = f(v)\sin\varphi\,, \quad z = g(v)$$

by v and φ. We now investigate when a curve $t \in \mathbb{R} \mapsto \mathbf{R}(t) \in \mathcal{S}$ is a geodesic. By definition the curve $\mathbf{R}$ is a geodesic if for all t

$$\ddot{\mathbf{R}}(t) \perp \mathcal{S}\,.$$

Comment. In the mechanical interpretation we look at a 'free particle' (a point mass of mass 1) moving over $\mathcal{S}$, i.e. without external forces like gravity. According to the d'Alembert principle, the point mass is kept on the surface $\mathcal{S}$ by the perpendicular force $\ddot{\mathbf{R}}(t)$.

1. Show that for a geodesic $t \in \mathbb{R} \mapsto \mathbf{R}(t) \in \mathcal{S}$ one has

$$\begin{aligned}\dot{\mathbf{R}} &= \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi + \dot{z}\mathbf{e}_z \\ \ddot{\mathbf{R}} &= (\ddot{r} - r\dot{\varphi}^2)\mathbf{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\mathbf{e}_\varphi + \ddot{z}\mathbf{e}_z\,.\end{aligned}$$

2. Show that $r^2\dot\varphi$ and $\frac{1}{2}\langle \dot{\mathbf{R}} \mid \dot{\mathbf{R}}\rangle = \frac{1}{2}(\dot r^2 + r^2\dot\varphi^2 + \dot z^2)$ are two (first) integrals of the system and that moreover
$$f'\ddot r \;-\; f' r\dot\varphi^2 \;+\; g'\ddot z \;=\; 0\,.$$
From now on we write $r(t) = f(v(t))$, $z(t) = g(v(t))$.

3. Show that the statements in item 2 are equivalent to
$$\begin{aligned} 2ff'\dot v\dot\varphi \;+\; f^2\ddot\varphi \;&=\; 0 \\ \ddot v \;-\; ff'\dot\varphi^2 \;&=\; 0\,. \end{aligned}$$

4. Show that from 3, in reverse, it follows that $\ddot{\mathbf{R}}(t) \perp \mathcal{S}$.

5. Define q_1, q_2, p_1 and p_2 by
$$q_1 \;=\; v\,,\; q_2 \;=\; \varphi\,,\; p_1 \;=\; \dot v\,,\; p_2 \;=\; f^2(v)\dot\varphi$$
and express $H = \frac{1}{2}\langle \dot{\mathbf{R}} \mid \dot{\mathbf{R}}\rangle$ in q_1, q_2, p_1 and p_2. Show that 3 is equivalent to the canonical form
$$\dot q_i \;=\; \frac{\partial H}{\partial p_i}\,, \quad \dot p_i \;=\; -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2)\,.$$
Now re-interprete the conservation laws found under 2.

6. Let $\theta = \theta(t)$ be the angle that the geodesic makes with the 'meridian'. Show that $|f\dot\varphi| = |\dot{\mathbf{R}}|\sin\theta$. Next show that $C = f\sin\theta$ is another first integral (this is the celebrated theorem of Clairaut).

7. Show that all meridians of $\mathcal{S}$ are geodesics and that a parallel circle $v = v_0$ of $\mathcal{S}$ is a geodesic precisely when $f'(v_0) = 0$.

8. Fix $p_2 = M$, taking $M \neq 0$. Reduce to one degree of freedom with the effective potential $V_M(q_1) = \dfrac{M^2}{2f^2(q_1)}$ (compare with the case of the central force field).
   (a) Show that if v_0 is a critical point, then the reduced system has an equilibrium $(q_1, p_1) = (v_0, 0)$. Compare with 7.
   (b) Describe the dynamics of the reduced system near such equilibria in the cases where v_0 is a maximum or a minimum.
   (c) Re-interprete the above findings for the original, unreduced system. Here describe the phase space and its decomposition in invariant level sets $p_2 = M$, $H = E$. What is the geometry of these sets and what is the corresponding dynamics? Also interprete the findings in the configuration space. Why is this description incomplete?

9. Explain the relationship of the items 1 - 5 with the calculus of variations.