Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 24 March.

7.1 Dissipative subsystems

Show that every system of n differential equations can be completed to a Hamiltonian system on $\mathbb{R}^{2n}$.

7.2 Linearization

Show that the linearization of a Hamiltonian system is again Hamiltonian. How are the corresponding Hamiltonian functions related to each other?

7.3 Area-preservation

Show that a planar dynamical system, defined on $\mathbb{R}^2$, is Hamiltonian if and only if it is area-preserving.

7.4 Conditional Liouville measure in the energy level

Given a symplectic manifold $(\mathcal{P}, \omega)$ and a (Hamiltonian) function $H : \mathcal{P} \longrightarrow \mathbb{R}$. Let $c \in \mathbb{R}$ be a regular value of H and consider the level $E_c := H^{-1}(c)$. Show that E_c is a manifold. Of what dimension? Also show that for any $x \in E_c$ the tangent space at E_c is given by $T_x E_c = \ker \mathrm{d}H_x$.

We consider the Liouville volume $2n$–form $\Omega := \omega \wedge \omega \wedge \cdots \wedge \omega$, the n–fold wedge product. It is known that the flow φ_t of the Hamiltonian vector field X_H preserves Ω. Also, the level E_c is preserved by φ_t. The present aim is to construct a 'conditional' volume Ω_c on E_c that is preserved by the restriction $\varphi_t | E_c$. So we consider $x \in E_c$ and tangent vectors $\xi_1, \xi_2, \ldots, \xi_{2n-1} \in T_x E_c$, having to define $\Omega_{c,x}(\xi_1, \xi_2, \ldots, \xi_{2n-1})$. To this end we write the equation

$$\Omega_x(\eta, \xi_1, \xi_2, \ldots, \xi_{2n-1}) \;=\; \mathrm{d}H_x(\eta) \cdot \Omega_{c,x}(\xi_1, \xi_2, \ldots, \xi_{2n-1})$$

where $\eta \in T_x \mathcal{P}$ is arbitrary. Show that this equation determines Ω_c in a unique way, independent of η. Also show that Ω_c is a nondegenerate $(2n-1)$–form, i.e. a volume form on E_c. Finally show that $\varphi_t | E_c$ preserves Ω_c.