Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 31 March.

8.1 Poisson and Lie brackets

In this exercise we study some properties of Poisson and Lie brackets on symplectic manifolds.

1. Show that if two functions F, G are integrals of H then so is $\{F, G\}$.
2. Show that if $\phi : \mathcal{P} \to \mathcal{P}$ is symplectic then $\{F, G\} \circ \phi = \{F \circ \phi, G \circ \phi\}$. What is the meaning of this equation?
3. Show that the map $F \mapsto X_F$ is a Lie algebra anti-homomorphism between the Lie algebra $(C^\infty(\mathcal{P}), \{,\})$ of smooth functions $\mathcal{P} \to \mathbb{R}$ and the Lie algebra $(\mathcal{F}^\infty(\mathcal{P}), [,])$ of smooth Hamiltonian vector fields on $\mathcal{P}$. In other words, show that $[X_F, X_G] = -X_{\{F,G\}}$ where the Lie bracket $[X, Y]$ of vector fields X, Y is the vector field defined by $[X, Y](F) = X(Y(F)) - Y(X(F))$ for any function $F : \mathcal{P} \to \mathbb{R}$.
4. Show that the Lie bracket of two locally Hamiltonian vector fields on a symplectic manifold is globally Hamiltonian. *Hint.* Use the identity $\iota_{[X,Y]}\alpha = \mathcal{L}_X \iota_Y \alpha - \iota_Y \mathcal{L}_X \alpha$.

8.2 A Hamiltonian system on $\mathbb{S}^2$

Until now we have seen only Hamiltonian systems on cotangent bundles T^*M. In this exercise we study a Hamiltonian system defined on $\mathbb{S}^2$ which is not a cotangent bundle.

In $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) consider the submanifold $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : x^2 = 1\}$ and the 2-form

$$\omega = x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2.$$

1. Show that ω is not closed. Show that ω is degenerate in $\mathbb{R}^3$ by finding at each point $x \in \mathbb{R}^3$ the space $N_x = \{\xi \in T_x\mathbb{R}^3 : \omega(\xi, -) = 0\}$. Show that the restriction $\varpi = \omega|_{\mathbb{S}^2}$ of ω to $\mathbb{S}^2$ is closed and non-degenerate. Show that ϖ is not exact.
2. Let $H : \mathbb{S}^2 \to \mathbb{R}$. Find the Hamiltonian vector field X_H on $\mathbb{S}^2$ that satisfies $\varpi(X_H, -) = dH(-)$.

3. Compute the Poisson brackets $\{x_i, x_j\}$, $i, j = 1, 2, 3$ with respect to ϖ and then compute the Poisson bracket $\{F, G\}$ for two arbitrary functions $F, G : \mathbb{S}^2 \to \mathbb{R}$.

4. Describe the dynamics of the Hamiltonian function $H = x_1$ on $\mathbb{S}^2$.

5. Show that every locally Hamiltonian vector field X on $\mathbb{S}^2$ is globally Hamiltonian, given that every boundaryless 1-chain on $\mathbb{S}^2$ is the boundary of a 2-chain.

Hint. Use appropriate coordinates on $\mathbb{S}^2$ (but be careful!).

8.3 Cotangent lifts

In the Lagrangian setting we can simplify the equations of motion by coordinate changes $q \mapsto Q = \phi(q)$. The corresponding transformations of the velocities and the momenta are determined by ϕ. In the Hamiltonian setting we can perform more general symplectic coordinate changes in phase space $\mathbb{R}^{2n}$. In this exercise we explore the relation between these two types of transformations.

1. Consider a mechanical system in $\mathbb{R}^n$ with coordinates $(q_1, \ldots, q_n)$ given by a Lagrangian function $L(q, \dot{q})$. For an arbitrary coordinate transformation $q \mapsto Q = \phi(q)$ compute the corresponding transformation of the momenta $p \mapsto P$. Show that the transformation $(q, p) \mapsto (Q, P) = \Phi(q, p)$ is symplectic with respect to the canonical symplectic form on $\mathbb{R}^{2n}$. The transformation Φ^{-1} is called the *cotangent lift* of ϕ.

2. Assume that there is an one-parameter group of diffeomorphisms $\phi^s : \mathbb{R}^n \to \mathbb{R}^n$ such that $L \circ \phi^s = L$. Show that the corresponding map $(q, p) \mapsto \Phi^s(q, p)$ is the flow of a Hamiltonian vector field X_F on $\mathbb{R}^{2n}$ and that $\{F, H\} = 0$ where H is the Legendre transform of L. Show that F is linear in p. What is the relation to Noether's theorem for Lagrangian systems?