Conservative Dynamical Systems

The last two exercises are homework, to be handed in on 26 May.

13.1 The simple pendulum

Consider the pendulum with Hamiltonian

$$H = \frac{1}{2}p^2 + (1 - \cos q).$$

1. Write H as a series
$$H = H_0 + H_1 + H_2 + \cdots,$$
by Taylor expanding H at the origin. Compute the series up to degree 6 terms.
2. Normalize H with respect to
$$H_0 = I = \frac{1}{2}(p^2 + q^2),$$
up to terms of degree 4. Express the normalized Hamiltonian in terms of I.
3. Compute the dependence on the energy E up to first degree terms of the frequency of periodic motions near the origin.

13.2 The 1:2:2 resonance

Consider in $\mathbb{R}^6$ a Hamiltonian $H = H_0 + H_1 + H_2 + \cdots$ that is a perturbation of the oscillator

$$H_0 = \frac{1}{2}(q_1^2 + p_1^2) + q_2^2 + p_2^2 + q_3^2 + p_3^2.$$

1. Show that if we normalize H with respect to H_0 then the degree 3 terms in the normal form are
$$\mathrm{Re}(a z_1^2 \bar{z}_2 + b z_1^2 \bar{z}_3),$$
where a, b are (complex) constants and $z_k = q_k + \mathrm{i} p_k$ for $k = 1, 2, 3$.

2. Show that there is a linear transformation $(z_1, z_2, z_3) \mapsto (w_1, w_2, w_3)$ such that the normal form up to degree 3 terms becomes

$$\mathcal{H} = \frac{1}{2} w_1 \bar{w}_1 + w_2 \bar{w}_2 + w_3 \bar{w}_3 + \mathrm{Re}(c w_1^2 \bar{w}_2),$$

where c is a constant that depends on a and b.

3. Show that $\mathcal{H}$ is a Liouville integrable Hamiltonian system by finding three integrals of motion.

13.3 The Hénon-Heiles system

Consider in $\mathbb{R}^4$ the Hamiltonian $H = H_0 + H_1$ where

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2),$$

and

$$H_1 = q_1^2 q_2 - \frac{1}{3} q_2^3.$$

1. Refer to Exercise 11.1. Why can you always express the normal form of H with respect to H_0 in terms of R, S, T and J ?

2. Show (without doing the normal form computation) that the normal form can contain no third degree terms and this is due only to the form of H_0 and not of H_1.

3. Use a software package like Mathematica to compute the normal form $\mathcal{H}$ of H with respect to H_0 up to fourth degree terms.

4. Consider the Poincaré surface of section Σ in $\mathbb{R}^4$ defined by $q_1 = 0$, $p_1 > 0$. For a fixed value of $H_0 = n$ compute the restriction of the normal form $\mathcal{H}$ on Σ. Draw (using a software package) the level curves of $\mathcal{H}$ on Σ for $n = \frac{1}{16}$.

5. Refer again to Exercise 11.1. Express $\mathcal{H}$ in terms of R, S, T and J. Determine the dynamics of $\mathcal{H}$ on the reduced space P_j.