Explicit Symmetries of the Kepler Problem

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The motion of a point mass $q(t)$ in the gravitation field of a mass that is fixed at the origin is described by the Kepler Hamiltonian

$$H(q,p) = \tfrac{1}{2}\|p\|^2 - \tfrac{\mu}{\|q\|}$$

Here, μ is determined by the gravitational constant and the two masses. Given an energy E, one regularizes the motion on the energy level hypersurface $\{(q,p)\,\big|\,H(q,p)=E\}$ by introducing the "eccentric anomaly" as new independent variable. If $E \neq 0$, the rescaled problem is the Hamiltonian flow of the Hamiltonian $\tilde{H}(q,p) = \frac{1}{\sqrt{2|E|}}\|q\|\big(\|p\|^2 - 2E\big) - \frac{2\mu}{\sqrt{2|E|}}$ on the hypersurface $\{(q,p)\,\big|\,\tilde{H}(q,p)=0\}$. See [CB], II.5.10b. Scaling q by the factor $\sqrt{2|E|}$ and p by the factor $\frac{1}{2|E|}$ allows to reduce the discussion to the case $2|E|=1$. That is, we consider the Hamiltonian flows of

$$K_\pm(q,p) = \|q\|\big(\|p\|^2 \pm 1\big) - 2\mu$$

on the hypersurfaces $\{(q,p)\,\big|\,K_\pm(q,p)=0\}$. The case of negative energy (that is, bounded orbits) is described by K_+, the case of positive energy by K_-.

It is well known that, in three dimensions, this problem has $SO(4)$ symmetry for negative energy and $SO_+(1,3)$ symmetry in the case of positive energy. We give explicit formulas for the actions of these groups on phase space $\mathbb{R}^3 \times \mathbb{R}^3$. Then we relate them to the regularizations of the Kepler problem by the Hopf map (Kustaanheimo–Stiefel [KS]) and by stereographic projection (Györgyi [Gy] /Moser [Mo] and Belbruno [Be]/Osipov [Os]).

To motivate the construction, we first consider the two dimensional situation. Here we identify phase space $\mathbb{R}^2\times\mathbb{R}^2$ with $\mathbb{C}\times\mathbb{C}$. The symmetry groups $SO(3)$ and $SO_+(1,2)$ have double covers $SU(2)$ and $SU(1,1)$, respectively. We show that the symmetries are related to the action of these groups on the momentum variable q by fractional linear transformations.

In three dimensions, we use the language of quaternions to describe the symmetries. In the case of negative energy, the symmetry group $SO(4)$ has as double cover $SU(2)\times SU(2)$, which may be viewed as the group of pairs of quaternions of norm one. We identify $\mathbb{R}^3$ with the set of all "pure quaternions". The action of the group then is given by a formula similar to the two dimensional situation. In the case of negative energy, the symmetry group $SO_+(1,3)$ has as double cover $SL(2,\mathbb{C})$. We describe $SL(2,\mathbb{C})$ in terms of quaternions and again give the formula for the action on phase space.

Then we select a point on the energy hypersurface $\{(q,p)\,\big|K_\pm(q,p)=0\}$. One gets a map from the symmetry group to the energy hypersurface by mapping a group element to the image of the chosen point under the action of the group element. We show that this map "is" the Kustaanheimo–Stiefel regularization. The regularizations of Györgyi/Moser and Belbruno/Osipov are based on the stereographic projections from the three dimensional sphere, respectively hyperboloid. By viewing the sphere as homogenous space for $SU(2)\times SU(2)$ and the hyperboloid as homogenous space for $SL(2,\mathbb{C})$, we get the relation between the Kustaanheimo–Stiefel regularization and the Györgyi/Moser resp. Belbruno/Osipov regularizations described by Kummer [Ku].

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