Dynamical Systems 2007

The last two exercises are homework, to be handed in on 12 February.

1.1 Hamiltonian phase portraits

Given a function $H : \mathbb{R}^2 \to \mathbb{R}$, of the form $H(p,q) = \frac{1}{2}p^2 + V(q)$. For several shapes of the graph of V, sketch the phase portrait of X_H. How do the integral curves intersect the q-axis? In particular consider cases where V has maxima, minima or a horizontal asymptote.

1.2 Linear Hamiltonian systems

What are the possible types of linear Hamiltonian systems? How do the corresponding Hamiltonian functions look like? Which of the occurring equilibria are structurelly stable — and in what sence?

1.3 Gradient- and Hamiltonian vector fields

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Consider the corresponding Hamiltonian vector field $X_H = \partial H/\partial y \; \mathrm{e}_1 - \partial H/\partial x \; \mathrm{e}_2$ and the gradient vector field $\mathrm{grad} H = \partial H/\partial x \; \mathrm{e}_1 + \partial H/\partial y \; \mathrm{e}_2$. Prove that, in each point of $\mathbb{R}^2$, the integral curves of X_H and $\mathrm{grad}H$ are orthogonal. What is the relation to the level curves of H ? Discuss the changes in the phase portraits of X_H and $\mathrm{grad}H$ if H is replaced by $-H$. In particular consider a neighbourhood of a minimum and a saddlepoint of H.

1.4 A bead on a wire, Huygens's isochronous curve

A bead with unit mass moves along a stiff wire, without friction. The wire lies in a vertical plane, the acceleration of gravity equals 1. Suppose the wire is given by the equation $y = U(x)$. Show that the system has energy $E = \frac{1}{2}\{1 + (dU/dx)^2\}\dot{x}^2 + U(x)$. Let q be an arclength parameter along the wire and put $p := \dot{q}$. Show that the system has the Hamiltonian form $\dot{p} = -\partial E/\partial q, \; \dot{q} = \partial E/\partial p$. Next assume that U attains a minimum at $x = x_0$. Prove that the frequency of 'small oscillations' at $x = x_0$ is equal to $\sqrt{U''(x_0)}$.

Subsequently we consider the special case where U is a cycloid, parametrically given by $\theta \mapsto (x,y) = (a(2\theta + \sin 2\theta), \; a(1 - \cos 2\theta))$. Here $a > 0$ is a constant while θ varies over the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Compute its arclength parameter and obtain its equations of motion. Prove that this device implements the harmonic oscillator.