Dynamical Systems 2007

These two exercises are homework, to be handed in on 16 April.

9.1 Conditional Liouville measure in the energy level

Given a symplectic manifold (M, ω) and a (Hamiltonian) function $H : M \to \mathbb{R}$. Let $c \in \mathbb{R}$ be a regular value of H and consider the level $E_c := H^{-1}(c)$. Show that E_c is a manifold. Of what dimension? Also show that for any $x \in E_c$ the tangent space at E_c is given by $T_x E_c = \ker dH_x$.

We consider the Liouville volume $2n$-form $\Omega := \omega \wedge \omega \wedge \cdots \wedge \omega$, the n-fold wedge product. It is known that the flow ϕ^t of the Hamiltonian vector field X_H preserves Ω. Also, the level E_c is preserved by ϕ^t. The present aim is to construct a 'conditional' volume Ω_c on E_c, that is preserved by the restriction $\phi^t | E_c$. So we consider $x \in E_c$ and tangent vectors $\xi_1, \xi_2, \cdots, \xi_{2n-1} \in T_x E_c$, having to define $\Omega_{c,x}(\xi_1, \xi_2, \cdots, \xi_{2n-1})$. To this end we write the equation

$$\Omega_x(\eta, \xi_1, \xi_2, \cdots, \xi_{2n-1}) \quad = \quad dH_x(\eta) \times \Omega_{c,x}(\xi_1, \xi_2, \cdots, \xi_{2n-1}) \ .$$

where $\eta \in T_x M$ is arbitrary. Show that this equation determines Ω_c in a unique way, independent of η. Also show that Ω_c is a nondegenerate $(2n-1)$ form, i.e. a volume form on E_c. Finally show that $\phi^t | E_c$ preserves Ω_c.

9.2 The harmonic oscillator revisited

Consider the harmonic oscillator with Hamiltonian function $H(p, q) = \frac{1}{2}p^2 + \frac{1}{2}\nu^2 q^2$, for $(p, q) \in \mathbb{R}^2$. As usual, X_H denotes the corresponding Hamiltonian vector field. Let $E > 0$ and consider the level curve $H^{-1}(E)$. By $T(E)$ we denote the period of oscillation in this level and by $A(E)$ the area enclosed by the level. Show that $A(E) = 2\pi E/\nu$ and $T(E) = 2\pi/\nu$. Show that by $q = \sqrt{2I/\nu} \cos\varphi,\ p = \sqrt{2\nu I} \sin\varphi$ a system (I, φ) of action-angle variables is defined for the oscillator. Give Hamiltonian function and the associated vector field in these coordinates.

Next compare with the previous exercise. Give the Liouville measure for the present problem. Also give an explicit expression for the conditional Liouville measure on the level curves of H.