Exercise 2. 4. 2015

1. Let $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function given by $H(x,y) = \frac{1}{2}y^2 + V(x)$ and assume that for $h \in \mathbb{R}$ the motion in the level set $H^{-1}(h)$ is periodic. Show that for energies near h the motion is also periodic. Let $A(h)$ denote the area enclosed by the level set $H^{-1}(h)$ and let $T(h)$ denote the period of the motion in this level set. Show that

$$T(h) \quad = \quad \left.\frac{\mathrm{d}A(z)}{\mathrm{d}z}\right|_{z=h}$$

and exemplify this for the harmonic oscillator with $V(x) = \frac{1}{2}x^2$. To this end, plot the level sets

$$H^{-1}(h) \quad = \quad \{\, (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 2h \,\}$$

for $h > 0$ and compute the area $A(h)$ of the region enclosed by this level set. Then compare the period with the rate of change of the area.