Exercise 4. 6. 2015

Consider the linear $\mathbb{S}^1$ action on phase space $\mathbb{R}^4$ given by

$$(t, \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix}) \quad \mapsto \quad \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{pmatrix} .$$

1. Compute the vector field X the flow of which is the above $\mathbb{S}^1$ action.
2. Show that the vector field X is the Hamiltonian vector field of the planar isotropic harmonic oscillator which is described by the Hamiltonian function
$$J(q,p) \;=\; \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) \; .$$
From now on we write $X = X_J$.
3. Introduce complex coordinates $z_k = q_k + \mathrm{i}p_k$, $k = 1, 2$ in $\mathbb{R}^4$. Express the above $\mathbb{S}^1$ action in terms of the coordinates z_1, z_2.
4. Define the polynomials $\pi_1 = z_1\bar{z}_1$, $\pi_2 = z_2\bar{z}_2$, $\pi_3 = \mathrm{Re}(z_1\bar{z}_2)$ and $\pi_4 = \mathrm{Im}(z_1\bar{z}_2)$. Show that π_k, $k = 1, 2, 3, 4$ are left invariant by the above $\mathbb{S}^1$ action.
5. Give an argument to show that the invariance of π_k with respect to the above $\mathbb{S}^1$ action implies that $\{\pi_k, J\} = 0$ (without computing directly the Poisson brackets).
6. Show that any polynomial that is invariant with respect to the above $\mathbb{S}^1$ action can be expressed through π_k. *Hint:* first try to determine the form of any arbitrary monomial in such invariant polynomial using complex coordinates.
7. Write $J = (\pi_1 + \pi_2)/2$ and $R = (\pi_1 - \pi_2)/2$, $S = \pi_3$, $T = \pi_4$. Show that $J^2 = R^2 + S^2 + T^2$.

According to the theory this implies that the reduced phase space $J^{-1}(j)/\mathbb{S}^1$ is diffeomorphic to the set

$$P_j \;=\; \left\{ \, (R, S, T) \in \mathbb{R}^3 \mid R^2 + S^2 + T^2 = j^2 \, \right\}$$

which is a 2–dimensional sphere for $j \neq 0$ and a single point for $j = 0$. The next question asks you to prove part of this fact.

8. Show that there is a bijective mapping between the set of orbits of X_J with fixed value $J = j$ and the points of P_j.

9. Recall that the reduced symplectic form ϖ_j is defined through the relation $\iota_j^* \omega = \rho_j^* \varpi_j$ where ρ_j is the reduction mapping

$$\begin{array}{rcccc} \rho_j & : & J^{-1}(j) & \longrightarrow & P_j \\ & & (q,p) & \mapsto & (R(q,p), S(q,p), T(q,p)) \end{array},$$

ι_j is the inclusion mapping

$$\begin{array}{rcccc} \iota_j & : & J^{-1}(j) & \longrightarrow & \mathbb{R}^4 \\ & & (q,p) & \mapsto & (q,p) \end{array},$$

and ω is the standard symplectic form

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$$

on $\mathbb{R}^4$. Compute the Poisson brackets between the quantities $J(q,p)$, $R(q,p)$, $S(q,p)$ and $T(q,p)$ in terms of (J, R, S, T).

10. Given a Hamiltonian function F on P_j, i.e., $F = F(j; S, R, T)$, use the Poisson brackets between (R, S, T) to derive the equations of motion. Show that the flow of *any* such F leaves P_j invariant, i.e. if an orbit starts on P_j then it stays there.

Consider in $\mathbb{R}^4$ the Hamiltonian $H = H_0 + H_1$ of the so-called Hénon-Heiles system where

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)$$

and

$$H_1 = q_1^2 q_2 - \frac{1}{3} q_2^3 .$$

11. Show (without doing the normal form computation) that the normal form can contain no third degree terms and this is due only to the form of H_0 and not of H_1.

12. Use a software package like Mathematica to compute the normal form $\mathcal{H}$ of H with respect to H_0 up to fourth degree terms.

13. Consider the Poincaré surface of section Σ in $\mathbb{R}^4$ defined by $q_1 = 0$, $p_1 > 0$. For a fixed value of $H_0 = n$ compute the restriction of the normal form $\mathcal{H}$ on Σ. Draw (using a software package) the level curves of $\mathcal{H}$ on Σ for $n = \frac{1}{16}$.

14. Express $\mathcal{H}$ in terms of R, S, T and J. Determine the dynamics of $\mathcal{H}$ on the reduced space P_j.