10

The Lagrange top

While all Hamiltonian systems are integrable in one degree of freedom, the passage to two degrees of freedom requires a second integral of motion for systems to be integrable. Noether's theorem 5.2 provides this second integral for those systems that admit a (continuous) symmetry group, but Theorem 9.1 shows that integrable systems are the exception rather than the rule. The qualitative jump from two to three degrees of freedom is not as sharp but again allows for new phenomena like Arnol'd diffusion. Rigid bodies help to find out what it means for a Hamiltonian system to be integrable in three degrees of freedom.

10.1 The phase space

A rigid body is a body in which the distances between all component particles remain fixed, therefore the shape is conserved. We speak of a heavy rigid body if it is fixed at one point and subject to a constant (gravitational) force. The Lagrange top is a rotationally symmetric heavy rigid body.

One can think of a heavy rigid body as a spherical pendulum that does not consist of a point mass, but of a mass distribution dm. The integral

$$m = \int dm$$

is the (total) mass of the rigid body; let us choose units in which this is scaled to $m = 1$. Then the first moments of the mass distribution are the components of the centre of mass while the second moments are related to the moments of inertia. The corresponding formulas are best written in a well-chosen co-ordinate system attached to the body.

Let (ξ_1, ξ_2, ξ_3) denote the co-ordinates with respect to an orthonormal set of axes $\{e_1, e_2, e_3\}$ attached to the body that has its origin at the fixed point. Then the centre of mass is given by $m_1 e_1 + m_2 e_2 + m_3 e_3$ with

$$m_i \quad = \quad \int \xi_i \, dm \ .$$

The second moments of the mass distribution form a symmetric 3 by 3 matrix

$$M \quad = \quad \left(\int \xi_i \xi_j \, dm \right)_{i,j=1,2,3}$$

which can be diagonalised, *i.e.* there is a rotation $T \in SO(3)$ such that TMT^{-1} is a diagonal matrix. Choosing $\{e_1, e_2, e_3\}$ to consist of eigenvectors of M the mixed moments vanish and the rotational symmetry of the Lagrange top lets two of the diagonal elements coincide. We choose these to be $M_1 = M_2$ whence e_3 is the symmetry or figure axis. In particular $m_1 = m_2 = 0$ since the centre of mass $m_3 e_3$ of a symmetric rigid body has to lie on the figure axis.

Next to the body set of axes we use a set of axes $\{e_x, e_y, e_z\}$ fixed in space that also has its origin at the fixed point. We arrange the vertical z–axis to be parallel to the gravitational force field, which however points downwards, so $G = -\gamma e_z$ with $\gamma > 0$. Then the potential energy of the Lagrange top reads

$$V \quad = \quad - \int \langle G \mid \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 \rangle \, dm \quad = \quad \chi \, \langle e_z \mid e_3 \rangle$$

with $\chi = \gamma m_3$, as if the total mass were concentrated at the centre of mass of the rigid body.

The fixed point of the rigid body prevents translational motions and only rotations about the fixed point are possible. Correspondingly, the configuration is determined by the matrix

$$g \quad = \quad \begin{pmatrix} \langle e_x \mid e_1 \rangle & \langle e_x \mid e_2 \rangle & \langle e_x \mid e_3 \rangle \\ \langle e_y \mid e_1 \rangle & \langle e_y \mid e_2 \rangle & \langle e_y \mid e_3 \rangle \\ \langle e_z \mid e_1 \rangle & \langle e_z \mid e_2 \rangle & \langle e_z \mid e_3 \rangle \end{pmatrix} \ . \tag{10.1}$$

Note that the potential energy $V = V(g) = \chi g_{33}$ is completely determined by the configuration of the Lagrange top; gravitation is a positional force field (and does not involve velocities).

Exercise 10.1. Show that the matrix (10.1) is an element of the group

$$SO(3) \quad = \quad \left\{ T \in M_{3\times 3}(\mathbb{R}) \;\middle|\; T^{\mathrm{T}} = T^{-1} \text{ and } \det T = 1 \right\}$$

of rotations (orientation preserving isometries) on $\mathbb{R}^3$ and specifies how to transform the spatial frame $\{e_x, e_y, e_z\}$ into the body set of axes $\{e_1, e_2, e_3\}$.

The kinetic energy is determined by the angular velocity, or, equivalently, by the angular momentum of the rigid body. For both we have the choice of measuring their components in the body frame or in the spatial frame. The equation

$$\ell_1 e_1 \;+\; \ell_2 e_2 \;+\; \ell_3 e_3 \;=\; \mu_1 e_x \;+\; \mu_2 e_y \;+\; \mu_3 e_z$$

relates the body co-ordinates $\ell \in \mathbb{R}^3$ of the angular momentum to the space co-ordinates $\mu \in \mathbb{R}^3$, and similarly

$$\omega_1 e_1 \;+\; \omega_2 e_2 \;+\; \omega_3 e_3 \;=\; \Omega_1 e_x \;+\; \Omega_2 e_y \;+\; \Omega_3 e_z$$

relates $\omega \in R^3$ to Ω for the angular velocity.

Exercise 10.2. Show that the configuration (10.1) relates body and space co-ordinates by means of $\mu = g(\ell)$ and $\Omega = g(\omega)$.

The kinetic energy is given by

$$\begin{aligned} T \;&=\; \frac{1}{2}\int \langle \omega \times \xi \mid \omega \times \xi \rangle \, dm \\ &=\; \frac{1}{2}\|\omega\|^2 \int \|\xi\|^2 \, dm \;-\; \frac{1}{2}\int \langle \omega \mid \xi \rangle^2 \, dm \\ &=: \; \frac{1}{2}\langle I\omega \mid \omega \rangle \end{aligned}$$

where the

$$I_{ij} \;=\; \delta_{ij}\int \xi_1^2 + \xi_2^2 + \xi_3^2 \, dm \;-\; \int \xi_i \xi_j \, dm$$

are the components of the symmetric matrix I. This matrix is the tensor of inertia of the rigid body, and our body frame is chosen to diagonalise, with principal moments of inertia $I_1 = M_2 + M_3 = M_1 + M_3 = I_2$ and $I_3 = M_1 + M_2$. Correspondingly, the axes along e_1, e_2 and e_3 are called the principal axes of inertia.

Exercise 10.3. Show that the tensor of inertia provides in body co-ordinates the relation $\ell = I\omega$ between angular momentum and angular velocity.

This allows to write the kinetic energy of the Lagrange top as

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \;=\; \frac{\ell_1^2 + \ell_2^2}{2I_1} \;+\; \frac{\ell_3^2}{2I_3} \;=\; \frac{\|\mu\|^2}{2I_1} \;-\; \frac{I_3 - I_1}{I_1 I_3} \cdot \frac{\ell_3^2}{2}$$

where we may choose to express the total (length of the) angular momentum $\|\ell\| = \|\mu\|$ both in body and space co-ordinates. The Hamiltonian

$$H \;=\; T \;+\; V$$

is a function on the phase space $T^*SO(3) \cong SO(3) \times \mathbb{R}^3$ that can be expressed not only in body co-ordinates (g, ℓ), but also in space co-ordinates (μ, g) since $\ell = g^{-1}(\mu)$. As shown in Appendix A the cotangent bundle $T^*SO(3)$ comes equipped with a natural Poisson structure. Here we first reduce a rotational symmetry before writing down the Poisson structure.

10.2 Rotational symmetries

The rotational symmetry of the Lagrange top makes the Hamiltonian function invariant under the S^1–action that rotates the body about the figure axis. Writing

$$\exp_\rho \quad := \quad \exp(\rho \cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}) \quad = \quad \begin{pmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

this S^1–action reads

$$\begin{array}{rccc} R : & S^1 \times (SO(3) \times \mathbb{R}^3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ & (\rho, (g, \ell)) & \mapsto & (g \circ \exp_\rho^{-1}, \exp_\rho(\ell)) \end{array}$$

in body co-ordinates. Since the group acts on $SO(3)$ by multiplication from the right one also speaks of the right S^1–action. Clearly the component ℓ_3 of the angular momentum along the figure axis is an invariant. To find all invariants it is convenient to express R in space co-ordinates where it becomes

$$\begin{array}{rccc} R : & S^1 \times (\mathbb{R}^3 \times SO(3)) & \longrightarrow & \mathbb{R}^3 \times SO(3) \\ & (\rho, (\mu, g)) & \mapsto & (\mu, g \circ \exp_\rho^{-1}) \end{array} \tag{10.2}$$

and reveals to leave invariant all three components of the angular momentum with respect to the spatial frame. Furthermore the components $\zeta_i := g_{i3}$ of the figure axis measured in the spatial frame are invariant under R. These invariants generate the ring of R–invariant functions on the phase space, for instance $\ell_3 = \langle \mu \mid \zeta \rangle$.

Since the gravitational force field is constant there is a second S^1–symmetry. Indeed, rotating the system about the vertical axis through the fixed point changes neither kinetic nor potential energy. This S^1–action corresponds to multiplying by $\exp_\rho$ from the left and is best expressed in body co-ortdinates where it reads

$$\begin{array}{rccc} L : & S^1 \times (SO(3) \times \mathbb{R}^3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ & (\rho, (g, \ell)) & \mapsto & (\exp_\rho \circ g, \ell) \end{array} \tag{10.3}$$

whence the ℓ_i and the components $\eta_i = g_{3i}$ of the vertical axis measured in the body frame generate the ring of L–invariant functions. The vector $\eta \in \mathbb{R}^3$ is called the Poisson vector.

Exercise 10.4. Express the left S^1–action L in space co-ordinates.

The two actions L and R commute and together define a 2–torus action

$$\begin{array}{rccc} L \cdot R : & \mathbb{T}^2 \times (SO(3) \times \mathbb{R}^3) & \longrightarrow & SO(3) \times \mathbb{R}^3 \\ & (\frac{1}{2\pi}\rho, (g, \ell)) & \mapsto & (\exp_{\rho_1} \circ g \circ \exp_{\rho_2}^{-1}, \exp_{\rho_2}(\ell)) \end{array}$$

on the phase space. Instead of reducing at once to one degree of freedom we reduce the symmetry in stages.

10.3 The Poisson structure

To reduce the right symmetry (10.2) we use the mapping

$$\begin{array}{rcccc} (\mu,\zeta) & : & SO(3)\times\mathbb{R}^3 & \longrightarrow & \mathbb{R}^3\times\mathbb{R}^3 \\ & & (g,\ell) & \mapsto & (g(\ell), g\begin{pmatrix}0\\0\\1\end{pmatrix}) \end{array}$$

expressing angular momentum and figure axis in the spatial frame. This enforces $\langle\zeta \mid \zeta\rangle = \|e_3\|^2 = 1$ for the reduced dynamics. The reduced Hamiltonian function of the Lagrange top is given by

$$H(\mu,\zeta) \;=\; \frac{\langle\mu \mid \mu\rangle}{2I_1} - \frac{I_3 - I_1}{I_1 I_3}\cdot\frac{\langle\mu \mid \zeta\rangle^2}{2} + \chi\cdot\zeta_3 \;.$$

We now derive the equations of motion. Since g is linear we infer from

$$\ell \;=\; I\omega \;=\; I_1\omega + (I_3 - I_1)\begin{pmatrix}0\\0\\\omega_3\end{pmatrix}$$

the expression

$$\Omega \;=\; \frac{1}{I_1}\mu - \frac{I_3 - I_1}{I_1}\omega_3\zeta$$

for the angular velocity in space co-ordinates. The equation $\dot{\xi} = \omega\times\xi$ for the velocity of every mass point on the rigid body in particular holds true for the centre of mass (or for the tip of the figure axis) and similarly translates to

$$\dot{\zeta} \;=\; \Omega\times\zeta \;=\; \frac{1}{I_1}\mu\times\zeta \;.$$

In the spatial frame the change of angular momentum results from the torque $e_z\times G$ whence

$$\dot{\mu} \;=\; \chi\begin{pmatrix}0\\0\\1\end{pmatrix}\times\zeta \;.$$

To cast these equations in Hamiltonian form we use the gradients

$$\begin{aligned} \nabla_\mu H &= \frac{1}{I_1}\mu - \frac{I_3 - I_1}{I_1 I_3}\langle\mu \mid \zeta\rangle\zeta \\ \nabla_\zeta H &= -\frac{I_3 - I_1}{I_1 I_3}\langle\mu \mid \zeta\rangle\mu + \chi\begin{pmatrix}0\\0\\1\end{pmatrix} \end{aligned}$$

and obtain

$$\begin{aligned} \dot{\mu} &= \nabla_\mu H\times\mu + \nabla_\zeta H\times\zeta \\ \dot{\zeta} &= \nabla_\mu H\times\zeta \end{aligned}$$

suggesting the Poisson bracket relations

$$\{\mu_i, \mu_j\} \;=\; \varepsilon_{ijk}\, \mu_k \;\;, \quad \{\mu_i, \zeta_j\} \;=\; \varepsilon_{ijk}\, \zeta_k \;\;, \quad \{\zeta_i, \zeta_j\} \;=\; 0$$

where $\varepsilon_{ijk} := \mathrm{sgn}\binom{1\,2\,3}{i\,j\,k}$ denotes the alternating Levi-Civita symbol. The inner products $\langle \mu \mid \zeta \rangle = \mu_1\zeta_1 + \mu_2\zeta_2 + \mu_3\zeta_3$ and $\langle \zeta \mid \zeta \rangle = \zeta_1^2 + \zeta_2^2 + \zeta_3^2$ are Casimir functions for this Poisson structure, ensuring in particular that the value 1 of the latter is conserved for all Hamiltonian dynamical systems on the reduced phase space. In case $\langle \mu \mid \zeta \rangle = 0$ we recover the spherical pendulum, but for $\langle \mu \mid \zeta \rangle \neq 0$ the figure axis is "loaded with angular momentum".

Exercise 10.5. Show that the reduced Lagrange top with fixed value $a = \langle \mu \mid \zeta \rangle$ of the component of the angular momentum along the figure axis is equivalent to the magnetic spherical pendulum considered in exercise 6.14. *Hint:* use the co-ordinate transformation $(\mu, \zeta) \mapsto (\mu \times \zeta, \zeta)$.

It was our choice to make e_3 the figure axis of the Lagrange top, so the Poisson structure on the phase space $T^*SO(3) \cong \mathbb{R}^3 \times SO(3)$ has to be preserved under right rotations about any axis. From this we derive the structure matrix

$$\begin{pmatrix}
0 & \mu_3 & -\mu_2 & 0 & g_{31} & -g_{21} & 0 & g_{32} & -g_{22} & 0 & g_{33} & -g_{23} \\
-\mu_3 & 0 & \mu_1 & -g_{31} & 0 & g_{11} & -g_{32} & 0 & g_{12} & -g_{33} & 0 & g_{13} \\
\mu_2 & -\mu_1 & 0 & g_{21} & -g_{11} & 0 & g_{22} & -g_{12} & 0 & g_{23} & -g_{13} & 0 \\
0 & g_{31} & -g_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g_{31} & 0 & g_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_{21} & -g_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{32} & -g_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g_{32} & 0 & g_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_{22} & -g_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & g_{33} & -g_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-g_{33} & 0 & g_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_{23} & -g_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

in the variables $(\mu_1, \mu_2, \mu_3, g_{11}, g_{21}, g_{31}, g_{12}, g_{22}, g_{32}, g_{13}, g_{23}, g_{33})$ on $\mathbb{R}^3 \times SO(3)$.

Exercise 10.6. Show that every Hamiltonian system on $\mathbb{R}^{12}$ (with the above Poisson strucure) leaves the submanifold $\mathbb{R}^3 \times SO(3)$ invariant. *Hint:* write down the Casimir functions.

Compared to the spherical pendulum the Lagrange top has the extra degree of freedom of rotation about the figure axis, and correspondingly the above structure matrix has rank 6. When expressed in body co-ordinates the Poisson bracket relations are similar. An important difference is the minus sign in

$$\{\ell_i, \ell_j\} \;=\; -\varepsilon_{ijk}\, \ell_k \tag{10.4}$$

which reflects that a counter-clock-wise rotation of the rigid body in space co-ordinates amounts to a clock-wise rotation of the spatial frame measured in the body set of axes.

Exercise 10.7. Show that reduction of the left S^1–action (10.3) yields the reduced equations of motion

$$\dot{\eta} = \eta \times \nabla_\ell H \tag{10.5a}$$
$$\dot{\ell} = \eta \times \nabla_\eta H + \ell \times \nabla_\ell H \tag{10.5b}$$

where η denotes the Poisson vector.

Note that the third principal moment of inertia I_3 does enter (10.5), while the right-reduced dynamics is the same for prolate, oblate and spherically symmetric Lagrange tops.

Exercise 10.8. Show that $\{\mu_3, \ell_3\} = 0$.

The functions μ_3 and ℓ_3 generate the left and right S^1–actions, respectively. Since the Hamiltonian is invariant under these group actions we know from Noether's theorem that $\{\mu_3, H\} = 0$ and $\{\ell_3, H\} = 0$. Where these conserved quantities are functionally independent the energy-momentum mapping

$$\begin{array}{rccc} \mathcal{EM} \ : & SO(3) \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (g, \ell) & \mapsto & (\ell_3, g_{31}\ell_1 + g_{32}\ell_2 + g_{33}\ell_3, H(g, \ell)) \end{array} \tag{10.6}$$

is regular and the common level sets

$$\mathcal{EM}^{-1}(a, b, h) \ = \ \{ \, \ell_3 = a \, , \, \mu_3 = b \, , \, H = h \, \}$$

form 3–dimensional invariant manifolds. The crucial point is that the flows of all three vector fields X_{ℓ_3}, X_{μ_3} and X_H commute and define an action of $\mathbb{T}^2 \times \mathbb{R}$ that is transitive on every connected component of $\mathcal{EM}^{-1}(a, b, h)$. Since the energy level sets $H^{-1}(h)$ are compact the connected components of $\mathcal{EM}^{-1}(a, b, h)$ are invariant 3–tori with conditionally periodic Hamiltonian flow generated by H.

Definition 10.1. A Hamiltonian system in n degrees of freedom is called integrable (in the sense of Liouville) if it admits n functionally independent real analytic integrals of motion in involution, i.e. having vanishing Poisson bracket relations.

The definition extends immediately to smooth conserved quantities F_i but then one has to explicitly specify that the subset of points in which the vector fiels X_{F_i} are linear independent is *e.g.* open, dense and of full phase space volume. The conclusion of conditionally periodic motions on the level sets $F^{-1}(a)$ can only be drawn for the compact connected components. In most cases one of the integrals of motion is chosen to be the Hamiltonian itself, in particular every Hamiltonian system in two degrees of freedom that admits a second integral of motion is integrable. An important class of examples are the normal forms in two degrees of freedom which provide an approximation that is automatically integrable. In higher degrees of freedom the normal form approximation near an elliptic equilibrium or periodic orbit need not be integrable.

10.4 Dynamics in one degree of freedom

The left S^1–action (10.3) induces the S^1–action

$$\begin{array}{rccc} L : & S^1 \times (\mathbb{R}^3 \times \mathbb{R}^3) & \longrightarrow & \mathbb{R}^3 \times \mathbb{R}^3 \\ & (\rho, (\mu, \zeta)) & \mapsto & (\exp_\rho(\mu), \exp_\rho(\zeta)) \end{array}$$

on the reduced phase space. Both the Poisson structure and the Hamiltonian are preserved under L. Again we use invariants to reduce the symmetry and take

$$\begin{aligned} \tau_1 &= \frac{\mu_1^2 + \mu_2^2}{2} \\ \tau_2 &= \frac{\zeta_1^2 + \zeta_2^2}{2} \\ \tau_3 &= \mu_1\zeta_1 + \mu_2\zeta_2 \\ \tau_4 &= \mu_1\zeta_2 - \mu_2\zeta_1 \\ \tau_5 &= \mu_3 \\ \tau_6 &= \zeta_3 \end{aligned}$$

as variables on the reduced phase space. From the first reduction we inherit the Casimir functions $\langle \mu \mid \zeta \rangle = \tau_3 + \tau_5\tau_6$ and $\langle \zeta \mid \zeta \rangle = 2\tau_2 + \tau_6^2$. Furthermore the generator $\mu_3 = \tau_5$ of L is a Casimir and from the syzygy we derive the fourth Casimir $S(\tau) = \frac{1}{2}(\tau_3^2 + \tau_4^2) - 2\tau_1\tau_2$. The invariants are restricted by the relations

$$\tau_1 \geq 0 \;, \quad \tau_2 \geq 0 \;, \quad S(\tau) = 0 \quad \text{and} \quad 2\tau_2 + \tau_6^2 = 1 \;.$$

Fixing $a = \tau_3 + \tau_5\tau_6$ and $b = \tau_5$ we can eliminate the variables

$$\tau_2 = \frac{1 - \tau_6^2}{2} \;, \quad \tau_3 = a - \tau_5\tau_6 \quad \text{and} \quad \tau_5 = b \;.$$

The remaining variables

$$x := \tau_6 = g_{33} \;, \quad y := \tau_4 = \mu_1 g_{23} - \mu_2 g_{13} \;, \quad z := \tau_1 = \frac{\|\mu\|^2 - b^2}{2}$$

are constrained by the relations

$$|x| \leq 1 \;, \quad z \geq 0 \quad \text{and} \quad S_a^b(x, y, z) = 0$$

with the remaining Casimir taking the form

$$S_a^b(x, y, z) = \frac{y^2}{2} - (1 - x^2)\, z + \frac{(a - bx)^2}{2} \;.$$

Exercise 10.9. Verify that the triple product

$$\{f, g\} \quad = \quad \langle \nabla f \times \nabla g \mid \nabla S_a^b \rangle$$

is the (twice) reduced Poisson structure.

The rank of the structure matrix being 2 the second step reduces from two to one degree of freedom. The reduced system is defined by the Hamiltonian function

$$H_a^b(x, y, z) \quad = \quad \frac{z}{I_1} \; + \; \chi x \; + \; \frac{a^2}{2I_3} \; + \; \frac{b^2 - a^2}{2I_1}$$

on the phase space

$$\mathcal{P}_a^b \quad = \quad \left\{ \; (x, y, z) \in \mathbb{R}^3 \; \middle| \; |x| \leq 1 \, , \; z \geq 0 \, , \; S_a^b(x, y, z) = 0 \; \right\} \quad .$$

Exercise 10.10. Perform the two reductions in the opposite order and show that the resulting Poisson spaces are diffeomorphic to the above $\mathcal{P}_a^b$. *Hint:* use the co-ordinate transformation $(x, y, z) \mapsto \big(x, \pm y, z + \frac{1}{2}(b^2 - a^2)\big)$.

The orbits in one degree of freedom are given by the intersections of $\mathcal{P}_a^b$ with the energy shells, the planes perpendicular to the vector

$$\begin{pmatrix} \chi \\ 0 \\ \frac{1}{I_1} \end{pmatrix} \quad ,$$

see Fig. **?**. In case $a \neq \pm b$ the syzygy $S_a^b(x, y, z) = 0$ prevents $x = \pm 1$ and we can divide this equation by $1 - x^2$ to obtain the explicit expression

$$z \quad = \quad z(x, y) \quad = \quad \frac{y^2 \; + \; (a - bx)^2}{2(1 - x^2)} \quad .$$

This makes $\mathcal{P}_a^b$ a graph over the (x, y)–space $]-1, 1[\, \times \mathbb{R}$. Intersections with the energy shells are periodic orbits that shrink down to a unique equilibrium; a conclusion which remains true for $a = -b$, with the regular elliptic equilibrium replaced by the singular equilibrium at $(x, y, z) = (-1, 0, 0)$.

Exercise 10.11. Prove the above assertion. *Hint:* show that $\mathcal{P}_a^b$ is convex.

For $a = b = 0$ we recover the spherical pendulum. The dynamics for $a = b$ depends on the exact common value taken by the components of the angular momentum along the figure axis e_3 and along the vertical axis e_z, see Fig. **?**.

From the periodic orbits on the reduced phase spaces $\mathcal{P}_a^b$ we reconstruct invariant 3–tori, superposing the rotation (spin) about the figure axis and the precession about the vertical axis. In particular, the inverse images $\mathcal{EM}^{-1}(a, b, h)$ of regular values of the energy-momentum mapping (10.6) are not only compact but also connected. Thus, the conditionally periodic motion of the Lagrange top consists of spin, precession and nutation, the latter being the periodic up and down of the figure axis that is encoded in the solution $x(t)$ of the one-degree-of-freedom system. Regular equilibria in one degree of freedom lead to invariant 2–tori with figure axis at constant height, this motion is called regular precession. The singular equilibria lead to periodic motions of the Lagrange top, in upright position if $a = b \neq 0$ and hanging down if $a = -b \neq 0$. The two invariant circles reconstructed from $(x, y, z) = (\pm 1, 0, 0)$ for $a = b = 0$ consist both of equilibria, differing from each other by the orientation of the body with respect to the vertical figure axis.

Exercise 10.12. Analyse the force free symmetric rigid body, or, equivalently, the Lagrange top with fixed centre of mass.

10.5 Gyroscopic stabilization

Instead of directly reconstructing the three-degrees-of-freedom dynamics from the reduced systems on $\mathcal{P}_a^b$ we can also reconstruct in stages and explicitly write down the corresponding motion in two degrees of freedom. The singular equilibria $(x, y, z) = (1, 0, 0)$ that exist for $a = b$ then give rise to regular equilibria $(\mu, \zeta) = (0, 0, a, 0, 0, 1)$ where a is the (fixed) value of the Casimir function $\ell_3 = \langle \mu \mid \zeta \rangle$. In the second reconstruction from two to three degrees of freedom these yield the family of periodic orbits parametrised by the common value a of μ_3 and ℓ_3.

For small $|a|$ this rotation (spin) in upright position is unstable, but becomes stable as $|a|$ gets sufficiently large, cf. Fig. **?**. The bifurcation value can be deduced from the one-degree-of-freedom system — for this value of $a = b$ the energy level set

$$\left\{ H_a^a \; = \; \chi + \frac{a^2}{2 I_3} \right\}$$

enters the conical singularity with the same "slope" as $\mathcal{P}_a^a$ at some direction of the cone. Note that for tangency of the energy shell to the reduced phase space $\mathcal{P}_a^a$ one must have $y = 0$. Consequently we have to compute the behaviour of H_a^a with respect to $\mathcal{P}_a^a$ within the (x, z)–plane. On $\mathcal{P}_a^a \cap \{y = 0\}$ and also on the energy shell we can write z as a function of x, and the relative behaviour of these two sets is given by the difference function

$$\Delta_a(x) \;\; = \;\; \frac{a^2}{2} \cdot \frac{1 - x}{1 + x} \; - \; I_1 \chi (1 - x) \;\; .$$

The bifurcation of the equilibrium $(x, y, z) = (1, 0, 0)$ takes place where the derivative $\Delta_a'(1) = -\frac{1}{4} a^2 + I_1 \chi$ vanishes, that is at $a = \pm 2 \sqrt{I_1 \chi}$.

It is easier to determine the mathematical type of the bifurcation in two degrees of freedom, where the bifurcating equilibrium $(\mu, \zeta) = (0, 0, a, 0, 0, 1)$ is regular (and not singular).

Exercise 10.13. Compute the eigenvalues of the linearization at $(\mu, \zeta) = (0, 0, a, 0, 0, 1)$ and show that for $a = \pm 2\sqrt{I_1 \chi}$ this equilibrium is in 1:−1 resonance.

The Krein collision of eigenvalues suggests that a Hamiltonian Hopf bifurcation takes place. The necessary conditions can again be checked in one degree of freedom, using the function Δ_a. The non-zero derivative

$$\frac{\mathrm{d}}{\mathrm{d}a} \Delta'_a(1) \bigg|_{a=\pm 2\sqrt{I_1 \chi}} = -\frac{a}{2} \bigg|_{a=\pm 2\sqrt{I_1 \chi}} = \mp \sqrt{I_1 \chi} \tag{10.7}$$

shows that we have linear versality (as could be deduced from the eigenvalue movement).

Next to this transversality condition we need to check non-degeneracy of the higher order terms. In normal form (7.8a) the coefficient b has to be non-zero. This ensures that at the bifurcation the energy shell has quadratic contact with the (conic) reduced phase space, and the sign of b distinguishes between the supercritical and the subcritical case. The non-zero derivative

$$\Delta''_a(1) \bigg|_{a=\pm 2\sqrt{I_1 \chi}} = \frac{a^2}{2} \bigg|_{a=\pm 2\sqrt{I_1 \chi}} = 2 I_1 \chi > 0 \tag{10.8}$$

shows that there is quadratic contact from the outside. We conclude that the Lagrange top undergoes supercritical Hamiltonian Hopf bifurcations as the common value a of μ_3 and ℓ_3 passes through $a = \pm 2\sqrt{I_1 \chi}$.

In three degrees of freedom the two phase space variables μ_3 and ℓ_3 are part of the energy-momentum mapping (10.6). The complicated geometry of the set of singular values of $\mathcal{EM}$ depicted in Fig. **?**. is prescribed by the two 1–parameter families of periodic orbits. The main part of this set is formed by a smooth surface parametrising the regular precessions. The spinning top hanging down is always stable, here the surface forms a crease. The surface also forms a crease where the spinning top in upright position is stable, *i.e.* for $a = b > 2\sqrt{I_1 \chi}$ and $a = b < -2\sqrt{I_1 \chi}$. Locally near the two Hamiltonian Hopf bifurcations the set of singular values of the energy-momentum mapping looks like the swallow tail surface (without the tail), in particular the line

$$(a, b, h) = \left(a, a, \chi + \frac{a^2}{2 I_3} \right)$$

detaches from the surface. The resulting 1–dimensional thread parametrises the unstable rotations of the Lagrange top in upright position; at $a = 0$ we recover the unstable equilibrium of the (spherical) pendulum.

With the exception of the thread the singular values form the boundary of the image of $\mathcal{EM}$. The regular values above this surface parametrise invariant 3–tori. From the spherical pendulum we inherit the non-existence of global action angle variables. As (a, b, h) reaches the thread, the 3–tori $\mathcal{EM}^{-1}(a, b, h)$

turn into a pinched torus, formed by the 1–dimensional hyperbolic periodic orbit and its 3–dimensional (coinciding) stable and unstable manifold.

The Lagrange top is an idealisation that is only approximatively realized. Neither is the body be perfectly symmetric, nor is the gravitational force field completely constant. What is persistent under the "small perturbation" from the "ideal" to the "real" system ?

The spinning top, both hanging down and in upright position, leads to periodic motions in the perturbed system as well; as long as the perturbation is sufficiently small we can apply the implicit mapping theorem. The Lagrange top has minimal energy $-\chi$ when hanging down without spinning. A small imperfection of the mass distribution is likely to break this S^1–symmetry and to single out an equilibrium where the Hamiltonian attains a strict minimum. The oscillatory motions about this minimum make clear that the perturbation cannot be "sufficiently small" for all rotations, but the faster the body rotates the more this rotation resembles the spinning of the Lagrange top.

Similarly we expect the S^1–symmetry of the equilibria in upright position to be broken. While these equilibria can be interpreted as an internal resonance of the periodic orbit, the Hamiltonian Hopf bifurcations correspond to a normal 1:−1 resonance. This bifurcation scenario does survive the perturbation (provided that the perturbation is sufficiently small, certainly much smaller than χ). One way to prove this is to normalize along the periodic orbit, thus introducing an S^1–symmetry that makes the third component of the angular momentum an integral of motion. Reducing this symmetry yields equilibria in two degrees of freedom to which the theory of Chapter 7 can be applied. The perturbation being sufficiently small the resulting quantities (10.7) and (10.8) will still be non-zero.

To the invariant 3–tori we want to apply the KAM-theory of Chapter 9. While tori too close to internal resonance are destroyed, those with Diophantine frequency vectors survive a sufficiently small perturbation. To make sure that indeed most tori are Diophantine we need to check that the unperturbed system (the Lagrange top) is non-degenerate, providing a local diffeomorphism from the parametrising actions to the frequency space. The non-existence of global action angle variables prevents that this can be done within a single co-ordinate system. Interestingly, the very reason for this non-existence allows to conclude that the non-degeneracy condition has to be fulfulled almost everywhere.

The regular precessions of the Lagrange top form a 2–parameter family of invariant 2–tori. Where these are internally resonant we cannot expect persistence. But here it is not enough that the two (internal) frequencies are Diophantine, we furthermore have to omit tori with normal-internal resonance. Indeed, the invariant 2–tori are reconstructed from elliptic equilibria and therefore have a normal frequency and we have seen in Chapter 8 how this can lead to bifurcations. Generalizing (9.6) to Diophantine conditions that also rule out normal-internal resonances one obtains a family of elliptic 2–tori parametrised by Cantor dust for which persistence can be proven.