Quasi-periodic stability of normally resonant tori.

Henk W. Broer[*], M. Cristina Ciocci[†], Heinz Hanßmann[‡] and André Vanderbauwhede[§]

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Abstract

We study quasi-periodic tori under a normal-internal resonance, possibly with multiple eigenvalues. Two non-degeneracy conditions play a role. The first of these generalizes invertibility of the Floquet matrix and prevents drift of the lower dimensional torus. The second condition involves a Kolmogorov-like variation of the internal frequencies and simultaneously versality of the Floquet matrix unfolding. We focus on the reversible setting, but our results carry over to the Hamiltonian and dissipative contexts.

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1 Introduction

Persistence results for quasi-periodic motions were first proved for maximal tori in Hamiltonian systems and became known as Kolmogorov–Arnol'd–Moser(KAM)theory. In [27] this was extended to lower dimensional tori and to other contexts like volume preserving and reversible systems. The rôle of the 'modifying terms' in terms of system parameters was clarified in [13, 21] and the Rüssmann condition [12, 31] allows to subsequently reduce the high number of parameters to the bare minimum.

These results yield what is called quasi-periodic (or normal linear) stability, i.e. families of invariant tori persist under sufficiently small perturbations when restricted to certain (measure-theoretically large) Cantor sets. The theorems in [12] make the crucial assumption that all eigenvalues of the matrix Ω describing the normal linear behaviour be simple. This implies in particular that $\det \Omega \neq 0$ (except for the disspative case and the high-dimensional volume preserving case, where this condition is explicitly added). Multiple resonances are admitted in [10, 15, 19] and the aim of the present paper is to admit zero eigenvalues without weakening the conclusion of quasi-periodic stability.

[*]Dept. of Mathematics and Computing Science, University of Groningen, POBox 407, 9700 AK Groningen, The Netherlands

[†]Dept. of Mathematics, Imperial College London, Campus South Kensington, Queen's Gate, London, SW72AZ, UK

[‡]Mathematisch Instituut, Universiteit Utrecht, Postbus 80.010, 3508 TA Utrecht, The Netherlands

[§]Dept. of Pure Mathematics and Computeralgebra, University of Gent, Krijgslaan 281, 9000 Gent, Belgium

1.1 Setting and results

We work on the phase space $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$, where $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ is the n-torus on which we use coordinates $x = (x_1, \ldots, x_n) \pmod{2\pi}$, while on $\mathbb{R}^m$ and $\mathbb{R}^{2p}$ we use respectively $y = (y_1, \ldots, y_m)$ and $z = (z_1, \ldots, z_{2p})$. In such coordinates a vector field on M takes the form

$$\dot{x} = f(x, y, z), \qquad \dot{y} = g(x, y, z), \qquad \dot{z} = h(x, y, z),$$

or in vector field notation:

$$X(x, y, z) = f(x, y, z)\partial_x + g(x, y, z)\partial_y + h(x, y, z)\partial_z. \tag{1.1}$$

We assume that the vector field X depends analytically on all variables, including possible parameters which we suppress for the moment; referring to [13, 21, 29] we note that our results remain valid when 'analyticity' is replaced by 'a sufficiently high degree of differentiability'. An invariant torus T of a vector field X is called parallel if a smooth conjugation exists of the restriction $X|_T$ with a constant vector field $\dot{x} = \omega$ on $\mathbb{T}^n$. The vector $\omega = (\omega_1, \omega_2, \ldots, x_n) \in \mathbb{R}^n$ is the (internal) frequency vector of T. The parallel torus is quasi-periodic when the frequencies are independent over the rationals.

We are concerned with persistence of quasi-periodic tori under small perturbations, and to fix thoughts we concentrate[1] on the reversible setting. To define reversibility we consider an involution (i.e. $G^2 = I$)

$$G : M \longrightarrow M, \quad (x, y, z) \mapsto (-x, y, Rz), \tag{1.2}$$

with $R \in \mathrm{GL}(2p, \mathbb{R})$ a linear involution on $\mathbb{R}^{2p}$ such that

$$\dim \mathrm{Fix}(R) = \dim \left\{ z \in \mathbb{R}^{2p} \mid Rz = z \right\} = p.$$

The vector field X is then called G-reversible (or reversible for short) if

$$G_*(X) = -X.$$

Using (1.1) this reversibility condition takes the explicit form

$$\begin{aligned} f(-x, y, Rz) &= f(x, y, z), \\ g(-x, y, Rz) &= -g(x, y, z), \\ h(-x, y, Rz) &= -Rh(x, y, z), \end{aligned}$$

valid for all $(x, y, z) \in M$.

Following [11, 12, 13, 21] the vector field X is called integrable if it is equivariant with respect to the group action

$$\mathbb{T}^n \times M \longrightarrow M, \quad (\xi, (x, y, z)) \mapsto (\xi + x, y, z)$$

of $\mathbb{T}^n$ on M, or in other words, if the functions f, g and h in (1.1) are independent of the x-variable(s). Such an integrable vector field

$$X(x, y, z) = f(y, z)\partial_x + g(y, z)\partial_y + h(y, z)\partial_z \tag{1.3}$$

[1] We give explicit formulations for reversible vector fields, but the results remain valid for e.g. dissipative, Hamiltonian or volume-preserving systems (vector fields and maps), where equivariance is also optional.

is reversible if

$$f(y,Rz) = f(y,z), \qquad g(y,Rz) = -g(y,z) \qquad \text{and} \qquad h(y,Rz) = -Rh(y,z) \tag{1.4}$$

for all $(y,z) \in \mathbb{R}^m \times \mathbb{R}^{2p}$; this implies $g(y,z) = 0$ for all $(y,z) \in \mathbb{R}^m \times \mathrm{Fix}(R)$. In case $h(0,0) = 0$ the[2] n-torus $T_0 = \mathbb{T}^n \times \{0\} \times \{0\}$ is invariant under the flow of the vector field X. The normal linear part $N(X)$ of (1.4) at T_0 is given by

$$N(X)(x,y,z) = \omega\partial_x + \Omega\, z\, \partial_z, \tag{1.5}$$

with

$$\omega = f(0,0) \quad \text{and} \quad \Omega = D_z h(0,0).$$

We denote the subspace of infinitesimally reversible linear operators on $\mathbb{R}^{2p}$ by $\mathfrak{gl}_-(2p;\mathbb{R})$ and by $\mathfrak{gl}_+(2p;\mathbb{R})$ the subspace of all R-equivariant linear operators on $\mathbb{R}^{2p}$, i.e.

$$\mathfrak{gl}_\pm(2p;\mathbb{R}) = \{\Omega \in \mathfrak{gl}(2p;\mathbb{R}) \mid \Omega R = \pm R\Omega\}.$$

In order to define the non-degeneracy of (1.3) at the invariant torus T_0 we consider the subspaces

$$\mathcal{X}^{\pm G}_{lin} = \big\{\omega\partial_x + \Omega z\partial_z \mid \omega \in \mathbb{R}^n, \Omega \in \mathfrak{gl}_\pm(2p;\mathbb{R})\big\}$$

of the spaces $\mathcal{X}^{-G}$ of all G-reversible vector fields on M and $\mathcal{X}^{+G}$ of all G-equivariant vector fields, satisfying $G_*(X) = +X$. For $X \in \mathcal{X}^{-G}$ the adjoint operator

$$\mathrm{ad}\, N(X) : \mathcal{X} \longrightarrow \mathcal{X}, \quad Y \mapsto [N(X), Y]$$

maps $\mathcal{X}^{\pm G}$ into $\mathcal{X}^{\mp G}$; a similar statement is true for $\mathcal{X}^{\pm G}_{lin}$.

Our interest concerns purely G-reversible vector fields, and G-reversible vector fields that are furthermore equivariant with respect to

$$F_l : M \longrightarrow M, \quad (x,y,z) \mapsto (x_1 - \frac{2\pi}{l}, x_*, y, z_I, \mathrm{e}^{\frac{2\pi \mathrm{i}}{l}} z_{II}). \tag{1.6}$$

Here $z_{II} \cong z_{2j-1} + \mathrm{i} z_{2j}$ singles out two of the z-variables in a complexified form and $z_I = (z_1, z_2, \ldots, z_{2j-2}, z_{2j+1}, \ldots, z_{2p})$ contains the remaining z-variables. To allow for a unified formulation of our results we define a reversing symmetry group Σ and a character (a group homomorphism) $\chi : \Sigma \longrightarrow \{\pm 1\}$ as follows:

(i) In the purely reversible case we set $\Sigma := \{\mathrm{Id}, G\}$ and $\chi(G) := -1$.

(ii) In the equivariant-reversible case we define Σ as the group generated by G and F_l and define χ by $\chi(G) := -1$ and $\chi(F_l) := 1$.

In both cases Σ is isomorphic to $\mathbb{Z}_2 \ltimes Z_l$, the dihedral group of order $2l$. When $l = 1$ the generator $F_1 = \mathrm{Id}$ of course is superfluous. For both cases we put

$$\begin{aligned} \mathcal{X}^+ &= \{X \in \mathcal{X} \mid E_*(X) = X \text{ for all } E \in \Sigma\} \\ \mathcal{X}^- &= \{X \in \mathcal{X} \mid E_*(X) = \chi(E)X \text{ for all } E \in \Sigma\} \end{aligned}$$

[2]Often one has a whole family $T_y = \mathbb{T}^n \times \{y\} \times \{0\}$ of invariant tori. While we are especially interested in bifurcations, the variable y will still act as a parameter, now unfolding the bifurcation scenario.

together with $\mathcal{X}_{lin}^{\pm} = \mathcal{X}_{lin}^{\pm G} \cap \mathcal{X}^{\pm}$. Furthermore we let $\mathcal{B}^{+}$ and $\mathcal{B}^{-}$ consist of the constant vector fields in $\mathcal{X}^{+}$ and $\mathcal{X}^{-}$, respectively and denote by

$$\mathcal{O}(\Omega_0) = \left\{ \mathrm{Ad}(A) \cdot \Omega_0 := A\,\Omega_0\,A^{-1} \mid A \in \mathrm{GL}_{+}(2p;\mathbb{R}) \right\}$$

the orbit under the adjoint action of $\mathrm{GL}_{+}(2p;\mathbb{R})$ on $\mathfrak{gl}_{-}(2p;\mathbb{R})$.

Definition 1 (Broer, Huitema and Takens [13]) *The parametrised*[3] *vector field* X_λ *with linearization* $N(X_\lambda)(x,y,z) = \omega(\lambda)\partial_x + \Omega(\lambda)z\partial_z$ *is* non-degenerate *at* $\lambda = \lambda_0 \in \mathbb{R}^s$ *if*

BHT(i) $\ker \mathrm{ad}\, N(X_{\lambda_0}) \cap \mathcal{B}^{+} = \{0\}$*;*

BHT(ii) *at* $\lambda = \lambda_0$ *the mapping* $(\omega, \Omega) : \mathbb{R}^s \longrightarrow \mathbb{R}^n \times \mathfrak{gl}_{-}(2p;\mathbb{R}), \lambda \mapsto (\omega(\lambda), \Omega(\lambda))$ *is transverse to* $\{\omega(\lambda_0)\} \times \mathcal{O}(\Omega(\lambda_0))$.

The two non-degeneracy conditions BHT(i) and BHT(ii) generalize the condition that $\mathrm{ad}\, N(X_{\lambda_0})$ has to be invertible, a requirement that lies at the basis of Mel'nikov's conditions ((1.7) with $|\ell| \neq 0$). One also speaks of BHT non-degeneracy. Compared to the formulation in [13], § 8a2 the requirement that $\Omega(\lambda_0)$ have only simple eigenvalues is dropped. The extension to multiple normal frequencies was developed in [10, 15, 19] for invertible $\Omega(\lambda_0)$; we return to the original formulation of BHT(i).

To formulate the strong non-resonance condition necessary for persistence of invariant tori we introduce for $\Omega \in \mathfrak{gl}_{-}(2p;\mathbb{R})$ the normal frequency mapping $\alpha : \mathfrak{gl}_{-}(2p;\mathbb{R}) \longrightarrow \mathbb{R}^{2p}$ where the components of $\alpha(\Omega)$ are equal to the imaginary parts of the eigenvalues of $\Omega \in \mathfrak{gl}_{-}(2p;\mathbb{R})$. Higher multiplicities are taken into account by repeating each eigenvalue as many times as necessary.

Definition 2 *A pair* $(\omega, \Omega) \in \mathbb{R}^n \times \mathfrak{gl}_{-}(2p;\mathbb{R})$ *is said to satisfy a* Diophantine condition *if there exist constants* $\tau > n-1$ *and* $\gamma > 0$ *such that*

$$|\ \langle k, \omega \rangle + \langle \ell, \alpha(\Omega) \rangle\ | \geq \gamma |k|^{-\tau} \tag{1.7}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ *and* $\ell \in \mathbb{Z}^{2p}$ *with* $|\ell| \leq 2$.

This condition is independent of the way in which we have ordered the components of $\alpha(\Omega)$; also, if (ω, Ω) satisfies (1.7) then the same is true for all $(\omega, \widetilde{\Omega})$ with $\widetilde{\Omega} \in \mathcal{O}(\Omega)$. For each $\Gamma \subset P$ we define the associated Diophantine subset

$$\Gamma_\gamma := \{ \lambda \in \Gamma \mid (\omega(\lambda), \Omega(\lambda))\, \text{satisfies (1.7)}\, \}.$$

When Γ is a small neighbourhood of some $\lambda_0 \in P$ where X is non-degenerate then Γ_γ is nowhere dense but with large measure (provided that γ is sufficiently small).

Theorem 3 *Let* $X \in \mathcal{X}^{-}$ *be a family of* Σ*-reversible integrable vector fields that is non-degenerate at* $\lambda_0 \in P$*. Then there exists* $\gamma_0 > 0$ *such that for all* $0 < \gamma < \gamma_0$ *the following*

[3] The rôle of the external parameter λ occurring in Definition 1 can be (partially) taken by the internal parameter y.

is true. There exists a neighbourhood Γ of λ_0, neighbourhoods $\mathcal{Y}$ and $\mathcal{Z}$ of the origin in respectively $\mathbb{R}^m$ and $\mathbb{R}^{2p}$, and a neighbourhood $\mathcal{U}$ of X in the compact-open topology on $\mathcal{X}^-$ such that for each $Z \in \mathcal{U}$ one can find a mapping $\Phi : \mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z} \times \Gamma \longrightarrow M \times P$ of the form

$$\Phi(x, y, z, \omega, \mu) = \Big(x + \widetilde{U}(x, \omega, \mu), y + \widetilde{V}(x, y, \omega, \mu), z + \widetilde{W}(x, y, z, \omega, \mu), \omega + \widetilde{\Lambda}_1(\omega, \mu), \mu + \widetilde{\Lambda}_2(\omega, \mu) \Big)$$

for which the following holds.

(i) *The mapping Φ is Σ-equivariant, real-analytic in the x-variable and normally affine in the y and z variables.*

(ii) *The mapping Φ is C^∞-close to the identity and is a C^∞-diffeomorphism onto its image.*

(iii) *The restriction of Φ to the Cantor set $\mathbb{T}^n \times \{0\} \times \{0\} \times \Gamma_\gamma$ of Diophantine X-invariant tori conjugates X to Z. The restiction of Φ to $\mathbb{T}^n \times \mathcal{Y} \times \mathcal{Z} \times \Gamma_\gamma$ also preserves the normal linear behaviour to these invariant tori.*

In terms of [12, 13], the conclusion of Theorem 3 expresses that the family X is quasi-periodically stable, i.e., structurally stable on a union of (Diophantine) quasi-periodic tori. This allows to condense Theorem 3 to the statement that non-degenerate Σ-reversible integrable vector fields are quasi-periodically[4] stable. Quasi-periodic stability implies that for every small perturbation Z there exists a Z-invariant 'Cantor set' $V \subset M \times P$ which is a C^∞-near-identity diffeomorphic image of the foliation $\mathbb{T}^n \times \{0\} \times \{0\} \times \Gamma_\gamma$ of n-tori. In the tori this diffeomorphism is an analytic conjugacy from X to Z, which also preserves the normal linear behaviour.

1.2 Normal-internal resonances

Resonances are at the core of the problems one has to solve when trying to prove quasi-periodic stability – persistence of elliptic invariant tori

$$T_y = \mathbb{T}^n \times \{y\} \times 0 \subseteq N := \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$$

under small perturbation. The strong non-resonance conditions (1.7) exclude in fact four types of resonances. An internal resonance

$$\langle k, \omega \rangle = 0 \quad \text{for some } 0 \neq k \in \mathbb{Z}^n$$

prevents the parallel flow on T_y to have a dense orbit whence the invariant torus is not a (minimal) dynamical object, but rather the union of closed invariant subtori. One cannot expect such an n-torus to persist, cf. [28, 32], for the same reason that a circle consisting of equilibria breaks up under perturbation (generically with only finitely many equilibria in the perturbed system). Such resonances are excluded by (1.7) when taking $\ell = 0$.

For $|\ell| = 1$ the inequalities (1.7) constitute the first Mel'nikov condition, cf. [3, 26, 35], and concern the normal-internal resonances

$$\langle k, \omega \rangle = \alpha_j \quad \text{with fixed } k \in \mathbb{Z}^n \text{ and } j \in \{1, \ldots, p\}. \tag{1.8}$$

[4] In [10] one speaks of 'normal linear stability' instead.

Passing to co-rotating co-ordinates on N yields this resonance with $k = 0$, cf. [9, 14]. This is a 2-step procedure. First k is brought into the form $k = (k_1, 0, \ldots, 0)$ by means of a preliminary transformation

$$N \longrightarrow N, \quad (x, y, z) \mapsto (\sigma x, y, z) \tag{1.9}$$

with $\sigma \in SL(n, \mathbb{Z})$. For the second step we again write $z_I = (z_1, z_2, \ldots, z_{2j-2}, z_{2j+1}, \ldots, z_{2p})$, $z_{II} = (z_{2j-1}, z_{2j})$ and complexify $z_{II} \cong z_{2j-1} + \mathrm{i}z_{2j}$. Then we perform a Van der Pol transformation

$$N \longrightarrow N, \quad (x, y, z) \mapsto (x, y, z_I, \mathrm{e}^{\mathrm{i}k_1 x_1} z_{II}). \tag{1.10}$$

The transformed vector field has a vanishing normal frequency $\alpha_j = 0$. Hence, already constant perturbations

$$\beta\partial_z = \beta_{2j-1}\partial_{z_{2j-1}} + \beta_{2j}\partial_{z_{2j}}, \quad \beta_{2j-1}, \beta_{2j} \in \mathbb{R}$$

make the tori T_y move in a way that cannot be compensated on the linear level. Condition BHT(i) in Definition 1 prevents such perturbations whence Theorem 3 yields quasi-periodic stability, see also Corollary 6 in Section 3. An alternative to this condition is to take (generic) higher order terms of the unperturbed vector field into account. This typically results in bifurcation scenarios that turn out to be quasi-periodically stable (in an appropriate sense) as well, cf. [5, 9, 17, 18, 33].

The remaining possibility $|\ell| = 2$ in (1.7) excludes the normal-internal resonances

$$\langle k, \omega \rangle = \alpha_i \pm \alpha_j \quad \text{with fixed } k \in \mathbb{Z}^n \text{ and } i \neq j \in \{1, \ldots, p\} \tag{1.11}$$

and

$$\langle k, \omega \rangle = 2\alpha_j \quad \text{with fixed } k \in \mathbb{Z}^n \text{ and } j \in \{1, \ldots, p\}. \tag{1.12}$$

For (1.11) one can again achieve $k = 0$ in co-rotating co-ordinates, cf. [35], turning this normal-internal resonance into the normal resonance

$$0 \neq \alpha_i = \pm\alpha_j, \quad i \neq j \in \{1, \ldots, p\}.$$

While now the invertibility of Ω does yield quasi-periodic stability of T_y, see [10, 15] and Corollary 4 in Section 3, the normal behaviour still may be drastically affected. Using the y-variable as a parameter, e.g. the normal linear matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

in a conservative setting unfolds (or deforms) both to elliptic and to hyperbolic behaviour. Here it is the bifurcation scenario involving the surrounding tori of dimension $n + 1$ and $n + 2$ that can only be captured by taking higher order terms of the unperturbed vector field into account; quasi-periodic stability was achieved in [7, 8, 18, 19] for the simplest conservative bifurcation scenarios.

The remaining case (1.12) is meaningful only if not already implied by (1.8), so assume that (1.7) holds with $|\ell| \leq 1$. Then we can still achieve $k = 0$ in co-rotating co-ordinates, but now on a 2-fold covering $M \longrightarrow N$ defined as follows. The preliminary transformation (1.9) brings the resonance vector k into the form $k = (k_1, 0, \dots, 0)$ with k_1 odd. The Van der Pol transformation is no longer a mapping from N to itself, but a covering mapping from $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$ onto N defined by

$$\Pi : M \longrightarrow N, \quad (x_1, x_*, y, z) \mapsto (2x_1, x_*, y, z_I, \mathrm{e}^{\mathrm{i}k_1 x_1} z_{II}).$$

Here $x_1 \in \mathbb{T}^1$ and $x_* = (x_2, \dots, x_n) \in \mathbb{T}^{n-1}$. The deck group $\mathbb{Z}_2 = \{\mathrm{Id}, F\}$ of this 2-fold covering is generated by the involution

$$F : M \longrightarrow M, \quad (x_1, x_*, y, z_I, z_{II}) \mapsto (x_1 - \pi, x_*, y, z_I, -z_{II}). \tag{1.13}$$

This means that

$$\Pi \circ F = \Pi.$$

Note that this is the special case $l = 2$ of (1.6), the corresponding quasi-periodic stability is stated in Corollary 5 of Section 3. The resulting frequency halving (or quasi-periodic period-doubling) bifurcation scenarios are quasi-periodically stable in the dissipative [5] and Hamiltonian [18] settings and similarly reversible frequency-halving bifurcations may be expected to occur if appropriate non-degeneracy conditions on the higher order terms are fulfilled.

In [3, 35][5] the second Mel'nikov condition ((1.7) with $|\ell| = 2$) is avoided completely, i.e. also simultaneous normal-internal resonances (1.11) and (1.12) with differing $k \in \mathbb{Z}^n$ are allowed. The price to pay for this approach is that any control on the linear behaviour is completely lost. For instance, double eigenvalues $\pm\mathrm{i}\alpha_1 = \pm\mathrm{i}\alpha_2$ generically unfold to a Krein collision, where an elliptic torus evolves a 4-dimensional normal direction of focus-focus type. Such changes cannot be captured without persistence of the (normal) linear behaviour.

1.3 Contents and conclusions

This paper fits in the framework of parametrised KAM theory [5, 7, 8, 9, 12, 18] that originates from Moser [27]; in fact we present a generalization of [10, 11, 13], as well as of [15, 19, 21]. In the next section we explicitly work out two examples to which Theorem 3 applies. In Section 3 we elaborate the conditions of Theorem 3 and also formulate three corollaries; the Corollaries 5 and 6 make the novel results of this paper explicit. The proof of Theorem 3 is sketched in Section 4. The necessary unfolding theory, which plays a key rôle, is deferred to the Appendix.

Our approach allows for normal-internal resonances (1.11) and (1.12) (and possibly also (1.8)) with $k \in \mathbb{Z}^n$ fixed. The ensuing deformations of the linear behaviour coming from the perturbation are taken care of by considering a versal unfolding of the linear part $\dot{z} = \Omega z$ of the unperturbed vector field, i.e., an unfolding that already contains all possible deformations. The necessary parameters are provided by $y \in \mathbb{R}^m$; the possibility that $m \geq n$ distinguishes the reversible context from the Hamiltonian setting. An alternative is to let the system depend on external parameters λ, where variation of (y, λ) versally unfolds the linear part.

The proof in Section 4 is formulated in terms of filtered Lie algebras and therefore exceeds the reversible setting, carrying over to other contexts that can be formulated in these terms, notably the dissipative, volume preserving and Hamiltonian contexts; possibly combined with

[5]These papers consider Hamiltonian systems, but we expect the results to carry over to the reversible context.

equivariance, cf. [10, 12]. In the Hamiltonian case this answers a conjecture formulated in [18] to the positive. For dissipative systems this has already been used in [5] when proving quasi-periodic stability of the frequency-halving bifurcation scenario. We expect that appropriate higher order terms in (3.9) allow to obtain a similar result for reversible systems.

The unfolding (A.8) recovers the result for the case $p = 2$ that was obtained in [22]. There a 4-dimensional reversible system with a codimension 2 singularity at the origin is studied by formal normal forms together with the persistence of the associated codimension 1 bifurcation phenomena. It would be interesting to investigate the persistence of the corresponding bifurcation scenario in the KAM setting. Note that an additional F-equivariance next to the G-reversibility would enforce the origin to be an equilibrium for the entire non-linear family, an assumption that is made in [22].

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2 Applications

We illustrate our results with two examples that explicitly show how our assumptions enter and what extra conclusions can be drawn.

Example 1 (Quasi-periodic response solutions) To show how to check the appropriate assumptions we consider the simple example of a 1-parameter family of quasi-periodically forced oscillators

$$\ddot{z} = f_\mu(t, z, \dot{z}) = h_\mu(t, \omega t, z, \dot{z}), \tag{2.1}$$

with a fixed frequency ω, for instance we take $\omega = \frac{1}{2}(\sqrt{5} - 1)$ (the golden mean number). The forcing h_μ is 2π-periodic in the first two arguments. The search is for quasi-periodic response solutions with this same frequency vector $(1, \omega)$.

Putting $z_1 = z$, $z_2 = \dot{z}$ we can rewrite (2.1) as an autonomous system

$$\begin{aligned}
\dot{x}_1 &= 1 \\
\dot{x}_2 &= \omega \\
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= h_\mu(x, z) = \bar{h}_\mu(z) + \tilde{h}_\mu(x, z)
\end{aligned}$$

on $\mathbb{T}^2 \times \mathbb{R}^2$ where we split h_μ into the average $\bar{h}_\mu$ over $\mathbb{T}^2 \times \{z\}$ and the oscillating part $\tilde{h}_\mu = h_\mu - \bar{h}_\mu$. The integrable vector field X_μ given by

$$\begin{aligned}
\dot{x}_1 &= 1 \\
\dot{x}_2 &= \omega \\
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \bar{h}_\mu(z)
\end{aligned}$$

has invariant 2-tori for all $z_1 \in \mathbb{R}$ with $\bar{h}_\mu(z_1, 0) = 0$. These correspond to response solutions of the forced oscillator.

Note that we allowed for h_μ to depend explicitly on z_2 whence $z_2 \mapsto -z_2$ is not a reversing symmetry. We impose the system to be reversible with respect to

$$(x_1, x_2, z_1, z_2) \mapsto (-x_1, -x_2, -z_1, z_2),$$

in particular $\bar{h}_\mu$ depends on z_1 only through z_1^2 and we concentrate on the invariant torus at $z = 0$. The dominant part

$$N(X_\mu) = \partial_{x_1} + \omega \partial_{x_2} + \Omega(\mu) z \partial_z$$

has the parameter-dependent 2×2 matrix

$$\Omega(\mu) = \begin{pmatrix} 0 & 1 \\ \partial_1 \bar{h}_\mu(0) & \partial_2 \bar{h}_\mu(0) \end{pmatrix}$$

which is invertible whenever $\partial_1 \bar{h}_\mu(0) \neq 0$. However, the non-degeneracy condition BHT(i) is also fulfilled if $\partial_1 \bar{h}_\mu(0) = 0$ since the eigenvector to the resulting eigenvalue 0 is not invariant under the involution

$$R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From this we conclude that condition BHT(i) is always satisfied.

The non-degeneracy condition BHT(ii) is satisfied when

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \partial_1 \bar{h}_\mu(0) \neq 0. \tag{2.2}$$

Thus, the system is BHT non-degenerate as soon as (2.2) holds true. Therefore, given this by Theorem 3 (for an explicit formulation see Corollary 6), if the oscillating part $\tilde{h}$ is sufficiently small, the forced oscillator (2.1) has a response solution near $z = 0$, with linear behaviour changing where $\partial_1 \bar{h}_\mu(0)$ passes through zero. □

Remarks.

(i) Earlier for the existence of a response solution as an extra requirement the condition $\partial_1 \bar{h}_\mu(0) \neq 0$ was needed [10, 11, 12, 26, 27].

(ii) The stability change of the response solution as $\partial_1 \bar{h}_\mu(0)$ passes through zero leads in the periodic case to additional periodic solutions bifurcating off from $z = 0$, cf. [24, 25, 30]. We expect such bifurcations to carry over to the quasi-periodic case.

We now return to the setting of the introduction where the normal-internal resonance (1.12) led to a perturbation problem on a 2:1 covering space. The next example shows how the normally linear vector fields on the covering and the base-space relate to one another.

Example 2 (Multiple normal-internal resonance) On the phase space

$$N = \mathbb{T}^2 \times \mathbb{R}^3 \times \mathbb{R}^4 = \{x, y, z\}$$

we consider the normally linear vector field

$$Y = 2\partial_{x_1} + \omega\partial_{x_2} + \Omega(\mu)z\partial_z$$

with

$$\Omega(\mu) = \begin{pmatrix} 0 & -1-\mu_1 & 1 & 0 \\ 1+\mu_1 & 0 & 0 & 1 \\ -\mu_2 & 0 & 0 & -1-\mu_1 \\ 0 & -\mu_2 & 1+\mu_1 & 0 \end{pmatrix}$$

where we think of the parameters $\nu = (\omega, \mu) \in \mathbb{R}^3$ as been obtained from $y \in \mathbb{R}^3$ by localization (3.7). The eigenvalues $\pm\mathrm{i}(1+\mu_1) \pm \sqrt{-\mu_2}$ of $\Omega(\mu)$ yield at $\mu = 0$ the normal frequency $\alpha = \pm\mathrm{i}$ that has two normal-internal resonances (1.11) and (1.12) with the same $k = (1, 0) \in \mathbb{Z}^2$. Complexifying both $\zeta_I \cong \zeta_1 + \mathrm{i}\zeta_2$ and $\zeta_{II} \cong \zeta_3 + \mathrm{i}\zeta_4$ on the covering space

$$\hat{N} = \mathbb{R}/(4\pi\mathbb{Z}) \times \mathbb{T} \times \mathbb{R}^3 \times \mathbb{R}^4 = \{\xi_1, \xi_2, \eta, \zeta\}$$

we have the covering mapping

$$\Pi : \hat{N} \longrightarrow N, \quad (\xi_1, \xi_2, \eta, \zeta) \mapsto (\xi_1 \mathrm{mod}(2\pi\mathbb{Z}), \xi_2, \eta, \mathrm{diag}[\mathrm{e}^{\frac{1}{2}\mathrm{i}\xi_1}]\,\zeta).$$

This leads to the deck transformation

$$F : \hat{N} \longrightarrow \hat{N}, \quad (\xi_1, \xi_2, \eta, \zeta) \mapsto (\xi_1 - 2\pi, \xi_2, \eta, -\zeta) \tag{2.3}$$

and the lifted vector field

$$\hat{Y} = \hat{\omega}_1\partial_{\xi_1} + \omega_2\partial_{\xi_2} + \hat{\Omega}(\mu)\zeta\partial_\zeta$$

on $\hat{N}$ satisfying $\Pi_*\hat{Y} = Y$. In this setting $\dot{\xi}_1 = \dot{x}_1$, implying that $\hat{\omega}_1 = 2$ and the corresponding periods are $\hat{T}_1 = 2\pi$ and $T_1 = \pi$, so $\hat{T}_1 = 2T_1$ as should be expected.

Regarding the Floquet matrices Ω and $\hat{\Omega}$ we have

$$\begin{aligned} \dot{z} &= \mathrm{diag}[\tfrac{1}{2}\mathrm{i}\dot{\xi}_1\mathrm{e}^{\frac{1}{2}\mathrm{i}\xi_1}]\,\zeta + \mathrm{diag}[\mathrm{e}^{\frac{1}{2}\mathrm{i}\xi_1}]\,\dot{\zeta} \\ &= \mathrm{diag}[\mathrm{e}^{\frac{1}{2}\mathrm{i}\xi_1}]\left(\frac{1}{2}\mathrm{i}\dot{\xi}_1\zeta + \hat{\Omega}\zeta\right) \\ &= \mathrm{diag}[\mathrm{e}^{\frac{1}{2}\mathrm{i}\xi_1}]\left(\mathrm{i}\,\mathrm{Id} + \hat{\Omega}\right)\zeta \\ &= \mathrm{diag}[\mathrm{e}^{\frac{1}{2}\mathrm{i}\xi_1}]\left(\mathrm{i}\,\mathrm{Id} + \hat{\Omega}\right)\mathrm{diag}[\mathrm{e}^{-\frac{1}{2}\mathrm{i}\xi_1}]\,z. \end{aligned}$$

Apparently

$$\Omega = \mathrm{diag}[\mathrm{e}^{\frac{1}{2}\mathrm{i}\xi_1}]\left(\mathrm{i}\,\mathrm{Id} + \hat{\Omega}\right)\mathrm{diag}[\mathrm{e}^{-\frac{1}{2}\mathrm{i}\xi_1}] = \mathrm{i}\,\mathrm{Id} + \hat{\Omega},$$

and the resulting family

$$\hat{\Omega}(\mu) = \Omega(\mu) - \mathrm{i}\,\mathrm{Id} = \begin{pmatrix} 0 & -\mu_1 & 1 & 0 \\ \mu_1 & 0 & 0 & 1 \\ -\mu_2 & 0 & 0 & -\mu_1 \\ 0 & -\mu_2 & \mu_1 & 0 \end{pmatrix}$$

of matrices is the LCU of

$$\hat{\Omega}(0) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Every perturbation of Y on N can be lifted to a perturbation of $\hat{Y}$ on $\hat{N}$ that respects the deck transformation (2.3) and rescaling time we can always arrange $\dot{x}_1 = 2$, i.e. that the first frequency equals 2. Applying Theorem 3 (for an explicit formulation see Corollary 5) we may conclude that $\hat{Y}$ is quasi-periodically stable and this implies quasi-periodic stability of Y. □

It should be noted that such an application of KAM Theory goes beyond the possibilities of [10, 15, 19]. For $n = 1$ the full bifurcation scenario has been addressed in [6] and it would be interesting to develop the extension from periodic to quasi-periodic orbits.

3 Main results

In the perturbation problem we work on the phase space $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p} = \{x, y, z\}$ where we are dealing with a 'dominant part'

$$\dot{x} = \omega, \quad \dot{y} = 0, \quad \dot{z} = \Omega z, \quad \text{or} \quad X = \omega\partial_x + \Omega z \partial_z \tag{3.1}$$

in vector field notation. While it is always possible to translate a single given invariant torus to $T_0 = \mathbb{T}^n \times \{0\} \times \{0\}$, it is an assumption on the system that this torus can be embedded in a whole family $T_y = \mathbb{T}^n \times \{y\} \times \{0\}$ of invariant tori parametrised by y. This can be equivalently stated as

$$h(y, 0) = 0 \quad \text{for all } y \in \mathbb{R}^m, \tag{3.2}$$

and the non-degeneracy condition BHT(i) in Definition 1 ensures that this assumption can be justified. For each $\epsilon > 0$ the scaling operator

$$\mathcal{D}_\epsilon : M \longrightarrow M, \; (x, y, z) \mapsto \left(x, \frac{y}{\epsilon}, \frac{z}{\epsilon^2}\right) \tag{3.3}$$

commutes with G and with the $\mathbb{T}^n$-action on M, and hence preserves reversibility and integrability. Using (1.3) and the linearity of $\mathcal{D}_\epsilon$ the push-forward $(\mathcal{D}_\epsilon)_* X$ of X under $\mathcal{D}_\epsilon$ takes the form

$$\begin{aligned} (\mathcal{D}_\epsilon)_* X(x, y, z) &= \mathcal{D}_\epsilon \left(X\big(\mathcal{D}_\epsilon^{-1}(x, y, z)\big)\right) \\ &= f(\epsilon y, \epsilon^2 z)\partial_x + \frac{1}{\epsilon} g(\epsilon y, \epsilon^2 z)\partial_y + \frac{1}{\epsilon^2} h(\epsilon y, \epsilon^2 z)\partial_z. \end{aligned}$$

We can use (1.4) to find that $N(X) := \lim_{\epsilon \to 0} (\mathcal{D}_\epsilon)_* X$ is the dominant part (1.5) of X. The vector field $N(X) = \omega\partial_x + \Omega z \partial_z$ is again reversible and integrable; it is characterised by the frequency vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ which describes the flow along the invariant tori T_y, and by the matrix $\Omega \in \mathfrak{gl}(2p;\mathbb{R})$ which determines the linear flow in the z-direction normal to the family of invariant tori.

Since Ω does not depend on the angular variable $x \in \mathbb{T}^n$ the vector field $N(X)$ is in normal linear Floquet form. The Floquet matrix Ω is infinitesimally reversible, satisfying $\Omega R = -R\Omega$ because of the reversibility of the vector field X. Observe that if $\mu \in \mathbb{C}$ is an eigenvalue of $\Omega \in \mathfrak{gl}_-(2p;\mathbb{R})$ then so is $-\mu$. Hence, the eigenvalues of Ω can be grouped into complex quartets, conjugate purely imaginary pairs $\pm i\alpha$, symmetric real pairs and the eigenvalue zero with even algebraic multiplicity.

3.1 Non-degeneracy conditions

Since $\mathrm{GL}_+(2p;\mathbb{R})$ is algebraic it follows that the orbit $\mathcal{O}(\Omega_0)$ is a smooth submanifold of $\mathfrak{gl}_-(2p;\mathbb{R})$. The tangent space at Ω_0 to this orbit is given by

$$T_{\Omega_0}\mathcal{O}(\Omega_0) = \{\mathrm{ad}(A)\cdot\Omega_0 = A\Omega_0 - \Omega_0 A \mid A \in \mathfrak{gl}_+(2p;\mathbb{R})\} = \mathrm{ad}(\Omega_0)\left(\mathfrak{gl}_+(2p;\mathbb{R})\right), \tag{3.4}$$

where we have used the fact that $\mathrm{ad}(A)\cdot\Omega = -\,\mathrm{ad}(\Omega)\cdot A$ for all $A, \Omega \in \mathfrak{gl}(2p;\mathbb{R})$. An unfolding of Ω_0 is a smooth mapping

$$\Omega : \mathbb{R}^s \longrightarrow \mathfrak{gl}_-(2p;\mathbb{R}),\ \mu \mapsto \Omega(\mu)$$

such that $\Omega(0) = \Omega_0$. An unfolding is versal if it is transverse to $\mathcal{O}(\Omega_0)$ at $\mu = 0$, which requires that $s \geq \operatorname{codim}\mathcal{O}(\Omega_0)$; a versal unfolding with the minimal number of parameters (i.e. with s equal to the codimension of $\mathcal{O}(\Omega_0)$ in $\mathfrak{gl}_-(2p;\mathbb{R})$) is called miniversal. Using the Implicit Function Theorem it is easily seen that given a miniversal unfolding $\Omega : \mathbb{R}^s \longrightarrow \mathfrak{gl}_-(2p;\mathbb{R})$ of $\Omega_0 \in \mathfrak{gl}_-(2p;\mathbb{R})$ we can write each $\widetilde{\Omega} \in \mathfrak{gl}_-(2p;\mathbb{R})$ near Ω_0 in the form $\widetilde{\Omega} = \mathrm{Ad}(A)\cdot\Omega(\mu)$ for some $(A,\mu) \in \mathfrak{gl}_+(2p;\mathbb{R}) \times \mathbb{R}^s$ close to $(\mathrm{Id}, 0)$ and depending smoothly on $\widetilde{\Omega}$. In case all eigenvalues of Ω_0 are different from each other a miniversal unfolding amounts to simultaneously deforming the eigenvalues, see [13]. Our approach yields persistence results independent of the eigenvalue structure of Ω_0 (see [34] for some other step towards such general persistence results). For more details on versal, miniversal (or universal) unfoldings we refer to [1, 2, 16].

Property BHT(i) generalizes the invertibility condition required in the definition of non-degeneracy as it was formulated in [10, 11, 15, 19]. What is really needed for the proofs is the invertibility of the linear operator

$$\mathrm{ad}\, N(X_{\lambda_0}) : \mathcal{B}^+ \longrightarrow \mathcal{B}^- \tag{3.5}$$

and since $\dim \mathrm{Fix}(R) = \dim \mathrm{Fix}(-R)$ this is fully captured by BHT(i). Computing

$$\mathrm{ad}\, N(X_{\lambda_0})(\beta\partial_z) = -\Omega(\lambda_0)\beta\partial_z \tag{3.6}$$

shows that this certainly holds true if $\det \Omega(\lambda_0) \neq 0$. However, the condition BHT(i) can still be satisfied if $\det \Omega(\lambda_0) = 0$, for example when $\ker(\Omega(\lambda_0)) \subset \mathrm{Fix}(-R)$. The Floquet matrix $\Omega(\lambda_0)$ may have zero eigenvalues as long as the corresponding eigenvectors do not lie in $\mathcal{B}^+$.

Remarks.

(i) Up to now, the condition $\det \Omega_0 \neq 0$ was one of the central assumptions for normal linear stability in the general dissipative context as well as in the volume preserving, symplectic and reversible contexts. Replacing this condition by BHT(i) allows to extend the known theorems to the singular case of eigenvalue zero.

(ii) Property BHT(i) is persistent under small variation of λ near λ_0 because of the upper-semi-continuity of the mapping $\lambda \mapsto \dim \ker \Omega(\lambda)$.

Property BHT(ii) means that locally the frequency vector $\omega(\lambda)$ varies diffeomorphically with λ, while 'simultaneously' the local family $\lambda \mapsto \Omega(\lambda)$ is a versal unfolding of $\Omega(\lambda_0)$ in the sense of [1, 2]. For earlier usage of this method in reversible KAM Theory, see [10, 11, 15]. In the Appendix we develop an appropriate versal unfolding that depends linearly on λ.

When trying to answer the persistence problem for T_y it is convenient to focus on (a sufficiently small neighbourhood of) each of the invariant tori T_ν ($\nu \in \mathbb{R}^m$) separately, considering the label $\nu \in \mathbb{R}^m$ of the chosen torus as a parameter; formally this can be done by a localizing transformation, setting

$$y = \nu + y_{loc} \qquad \text{and} \qquad X_{loc}(x, y_{loc}, z; \nu) := X(x, \nu + y_{loc}, z). \tag{3.7}$$

This way we get a parametrised family of reversible and integrable vector fields, still on the same state space M; in this localized situation we concentrate on the persistence in a small neighbourhood of the invariant torus T_0, corresponding to $(y_{loc}, z) = (0, 0)$. For simplicity we absorb the additional parameter ν with the other parameters which we may have and we also drop the subscript 'loc'.

The non-degeneracy condition BHT(ii) requires that $s \geq n + \operatorname{codim} \mathcal{O}(\Omega(\lambda_0))$; in case all parameters originate from a localization procedure this means that we should have $m \geq n + \operatorname{codim} \mathcal{O}(\Omega(\lambda_0))$. Assume now that X_λ is non-degenerate at $\lambda_0 \in \mathbb{R}^s$, and let $(\omega_0, \Omega_0) := (\omega(\lambda_0), \Omega(\lambda_0))$. Using a re-parametrisation and a parameter-dependent linear transformation in the z-space we may assume that the parameter λ takes the form $\lambda = (\omega, \mu, \tilde{\mu})$ and belongs to a neighbourhood P of $\lambda_0 := (\omega_0, 0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{s-n-c}$, while the dominant part of the vector field reads

$$N(X)(x, y, z, \omega, \mu, \tilde{\mu}) = \omega \partial_x + \Omega(\mu) z \partial_z, \tag{3.8}$$

where $\Omega : \mathbb{R}^c \longrightarrow \mathfrak{gl}_-(2p; \mathbb{R})$ is a given miniversal unfolding of Ω_0. The $\tilde{\mu}$-part of the parameter does not appear in this expression for the (unperturbed) vector field X; although it might appear explicitly in the perturbations it turns out that $\tilde{\mu}$ plays no role at all in the further analysis. Therefore we suppress $\tilde{\mu}$ and just keep the essential parameters (ω, μ) and set $s = n + c$, with $c = \operatorname{codim} \mathcal{O}(\Omega(0))$. The question of a particular choice for the miniversal unfolding $\Omega(\mu)$ appearing in (3.8) is addressed in the Appendix.

3.2 Consequences

We are given a family of integrable vector fields

$$X(x, y, z, \omega, \mu) = [\omega + f(y, z, \omega, \mu)]\, \partial_x + g(y, z, \omega, \mu)\, \partial_y + [\Omega(\mu) z + h(y, z, \omega, \mu)]\, \partial_z \tag{3.9}$$

on the product $M \times P$ of phase space $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$ and parameter space $P \subseteq \mathbb{R}^s = \mathbb{R}^n \times \mathbb{R}^c$ with reversing symmetry group Σ generated by (1.2) and (1.6). For $l = 1$ the latter is just the identity, but for $l \geq 2$ the composition

$$H_l := F_l \circ G : M \longrightarrow M, \quad (x, y, z) \mapsto (\frac{2\pi}{l} - x_1, x_*, y, S_l Rz), \tag{3.10}$$

is another reversing symmetry and one may also characterise the vector fields in $\mathcal{X}^-$ as being reversible with respect to the two mappings G and H_l. Note that in this characterisation H_l may be replaced by $F_l^i \circ G$ for any i relative prime to l.

The coefficient functions f, g and h entering X are higher order terms in z, satisfying $f(y, 0, \omega, \mu) = g(y, 0, \omega, \mu) = h(y, 0, \omega, \mu) = D_z h(y, 0, \omega, \mu) = 0$ for all $y \in \mathbb{R}^m$, $(\omega, \mu) \in P$. Within $\mathcal{X}^-$ we consider perturbations Z of X and write

$$Z(x, y, z, \omega, \mu) = \Big[\omega + \tilde{f}(x, y, z, \omega, \mu)\Big]\, \partial_x + \tilde{g}(x, y, z, \omega, \mu)\, \partial_y + \Big[\Omega(\mu) z + \tilde{h}(x, y, z, \omega, \mu)\Big]\, \partial_z;$$

here the coefficient functions $\tilde{f}, \tilde{g}$ and $\tilde{h}$ may contain lower order terms but are close to f, g and h, respectively. In this situation Theorem 3 allows to conjugate Z to X as far as Diophantine tori are concerned.

The condition that Φ be a full conjugation from X to Z means that $\Phi_*(X) = Z$, or equivalently $\left(\Phi^{-1}\right)_* (Z) = X$. What is actually proved is the existence of a local diffeomorphism Φ such that

$$\left(\Phi^{-1}\right)_* (Z)(x, y, x, \omega, \mu) = N(X)(x, y, z, \omega, \mu) + O(|y|, |z|)\partial_x + O(|y|, |z|^2)\partial_y + O(|y|, |z|^2)\partial_z \tag{3.11}$$

for all $(\omega, \mu) \in P_\gamma$ which are sufficiently close to $(\omega_0, 0)$. The property (3.11) implies that for all parameter values (ω, μ) in the indicated Cantor set the X-invariant torus $\mathbb{T}^n \times \{0\} \times \{0\}$ is mapped by Φ into a Z-invariant torus on which the Z-flow is conjugate to the constant flow $\omega\partial_x$ on $\mathbb{T}^n$. This means that a Cantor subset of large measure of the family $\mathbb{T}^n \times \{0\} \times \{0\} \times P$ of X-invariant tori survives the perturbation to Z. The preservation of the normal linear behaviour means that the normal linear vector fields $N(X)$ and $N(Z)$ along two corresponding invariant tori are conjugated by the derivative of the C^∞-near-identity diffeomorphism.

In comparison to earlier results on persistence of lower-dimensional tori the condition that all eigenvalues be simple is dropped in Theorem 3 and the condition $\det \Omega(0) \neq 0$ is weakened to BHT(i). Indeed, we have the following corollary.

Corollary 4 (Ciocci [15], Broer, Hoo and Naudot [10]) *Let the family $X \in \mathcal{X}^-$ of G-reversible integrable vector fields satisfy the non-degeneracy condition* BHT(ii) *at $\lambda_0 = (\omega_0, 0) \in P$, with $\Omega(0)$ invertible. Then X is quasi-periodically stable.*

Next to the above purely reversible case $l = 1$ also the case $l = 2$ of a reversing symmetry group $\Sigma = \{\mathrm{Id}, F, G, H\}$ merits an explicit formulation. Here $H = H_2$ is given by (3.10) and yields

$$f(y, SRz) = f(y, z), \qquad g(y, SRz) = -g(y, z) \qquad \text{and} \qquad h(y, SRz) = -SRh(y, z) \tag{3.12}$$

for integrable vector fields where $S(z_I, z_{II}) = (z_I, -z_{II})$. From S-equivariance it follows that f, g are even in z_{II}, while h is odd in z_{II}. Moreover, (1.4) and (3.12) imply that $g(y, z) = 0$ for all $(y, z) \in \mathbb{R}^m \times \mathrm{Fix}(R)$ and also for all $(y, z) \in \mathbb{R}^m \times \mathrm{Fix}(SR)$.

Corollary 5 *Let $X \in \mathcal{X}^-$ be a family of G-reversible F-equivariant integrable vector fields that satisfies the non-degeneracy condition* BHT(ii) *at $\lambda_0 = (\omega_0, 0) \in P$, with $\Omega(0)$ invertible on* $\mathrm{Fix}(S)$*. Then X is quasi-periodically stable.*

Again we allow for multiple eigenvalues, in particular the eigenvalue 0 may have multiplicity larger than two. A similar statement holds in case of equivariance with respect to (1.6) instead of (1.13).

In the covering setting of Section 1, we observe that the lift of an integrable vector field again is integrable. In fact, if $\hat{X}$ is the lift to M of an integrable vector field X on N, then $\Pi_*(\hat{X}) = X$ and $F_*\hat{X} = \hat{X}$. In case the second Mel'nikov condition is violated by a resonance (1.12) we can apply Corollary 5 on a 2:1 covering space. In Example 2 of Section 2 we do this for a double normal-internal resonance with fixed resonance vector $k \in \mathbb{Z}^2$.

Corollary 6 *Let $X \in \mathcal{X}^-$ be a family of G-reversible integrable vector fields that satisfies the non-degeneracy condition* BHT(ii) *at $\lambda_0 = (\omega_0, 0) \in P$. If* $\ker \Omega(0)$ *is contained in* $\mathrm{Fix}(-R)$ *then X is quasi-periodically stable.*

If $\ker \Omega(0) \subseteq \mathrm{Fix}(+R)$ we generically expect a quasi-periodic centre-saddle bifurcation to take place, cf. [17]. Here violations of the first Mel'nikov condition prevents persistence of the corresponding tori if not embedded in an appropriate bifurcation scenario. The scaling (3.3) also can be applied to non-integrable systems, making the non-integrable higher order terms a small perturbation. It is then not automatic that the resulting dominant part is in Floquet form, this is a necessary extra requirement that can be thought of as generalization of integrability under which quasi-periodic stability can still be achieved. For a more thorough discussion of these questions see [13].

4 Sketch of proof

The proof of Theorem 3 follows [4, 10, 13] almost verbatim (see also [11, 15, 19]) and here we just concentrate on the novel aspects. The quite universal set-up of [13, 27] is based on a Lie algebra approach, using a standard Newtonian linearization procedure. The conjugation Φ between the integrable and the perturbed family is produced as the limit of an infinite iteration process. The central ingredient of the proof is the solution of the linearized problem, the so-called homological equation. The structure at hand, that is, the reversible symmetry group Σ, is phrased in terms of the Lie algebras $\mathcal{X}^\pm$, $\mathcal{X}^\pm_{lin}$ and $\mathcal{B}^\pm$ and therefore automatically preserved. Here we content ourselves showing how the non-degeneracy conditions BHT(i) and BHT(ii) enter when solving the homological equation.

At each iteration step we look for a transformation $(\xi, \eta, \zeta, \sigma, \nu) \mapsto (x, y, z, \omega, \mu)$ with $\omega = \sigma + \Lambda_1(\sigma, \nu)$ and $\mu = \nu + \Lambda_2(\sigma, \nu)$ independent from the variables (ξ, η, ζ) so that the projection to the parameter space P is preserved. The transformation in the variables is generated by a Σ-equivariant vector field $\Psi \in \mathcal{X}^+$ that we write as

$$\Psi = U\partial_x + V\partial_y + W\partial_z.$$

The homological equation reads

$$\operatorname{ad} N(X)(\Psi) = L + N \tag{4.1}$$

with

$$L_{\sigma,\nu}(\xi,\eta,\zeta) = \{Z - X\}_{lin,d} \quad \text{and} \quad N_{\sigma,\nu}(\xi,\eta,\zeta) = \Lambda_1(\sigma,\nu)\partial_\xi + \Omega(\Lambda_2(\sigma,\nu))\zeta\partial_\zeta$$

and determines the unknown U, V, W, Λ_1 and Λ_2 according to

$$\begin{aligned} U_\xi\sigma &= \Lambda_1 + \widetilde{f} \\ V_\xi\sigma + V_\zeta\Omega(\nu)\zeta &= \widetilde{g} + \widetilde{g}_\eta\eta + \widetilde{g}_\zeta\zeta \\ W_\xi\sigma + [\Omega(\nu)\zeta, W] &= \Omega(\Lambda_2)\zeta + \widetilde{h} + \widetilde{h}_\eta\eta + \widetilde{h}_\zeta\zeta. \end{aligned} \tag{4.2}$$

Here $U, V, W, \widetilde{f}, \widetilde{g}, \widetilde{h}$ and their derivatives depend on $(\xi, 0, 0, \sigma, \nu)$. Moreover, greek subscripts denote derivatives, while $U_\xi\sigma = \Sigma_{j=1}^n U_{\xi_j}\sigma_j$ and similarly for V and W. These linear equations are solved by suitably truncated Fourier series. Note that the left hand side of (4.2) consists of the components of the vector field $\operatorname{ad} N(X_{\sigma,\nu})(\Psi)$, where

$$N(X_{\sigma,\nu})(\xi,\eta,\zeta) = \sigma\partial_\xi + \Omega(\nu)\zeta\partial_\zeta.$$

For a given Z (and hence L), the goal is to find $\Psi \in \mathcal{X}^+_{lin,d}$ and $N \in \ker \operatorname{ad} N(X)^T \subseteq \mathcal{X}^+_{lin,d}$ so that the homological equation (4.1) is satisfied. Here $\mathcal{X}^+_{lin,d} = \mathcal{X}^+_{lin} \cap \mathcal{X}^+_d$ denotes the intersection set of the Taylor and Fourier truncations of vector fields in $\mathcal{X}^+$.

We make the ansatz

$$V(\xi,\eta,\zeta,\sigma,\nu) = V_0 + V_1\eta + V_2\zeta \quad \text{and} \quad W(\xi,\eta,\zeta,\sigma,\nu) = W_0 + W_1\eta + W_2\zeta \tag{4.3}$$

for the unknown Ψ, where V_j and W_j $(j = 0, 1, 2)$ depend on ξ and on the multiparameter (σ, ν). Fourier expanding in ξ and comparing coefficients in (4.2) yields the following equations for an explicit (formal) construction of Ψ. To avoid clumsy notation we suppress the dependence on (σ, ν).

For $k \neq 0$, equation (4.2) implies

$$\mathrm{i}\langle k,\sigma\rangle U_k = \widetilde{f}_k \tag{4.4}$$
$$\mathrm{i}\langle k,\sigma\rangle V_{0,k} = \widetilde{g}_k, \tag{4.5}$$
$$\mathrm{i}\langle k,\sigma\rangle V_{1,k} = (\widetilde{g}_\eta)_k \tag{4.6}$$
$$V_{2,k}\,[\mathrm{i}\langle k,\sigma\rangle \operatorname{Id} + \Omega(\nu)] = (\widetilde{g}_\zeta)_k \tag{4.7}$$
$$[\mathrm{i}\langle k,\sigma\rangle \operatorname{Id} - \Omega(\nu)]\, W_{0,k} = \widetilde{h}_k \tag{4.8}$$
$$[\mathrm{i}\langle k,\sigma\rangle \operatorname{Id} - \Omega(\nu)]\, W_{1,k} = (\widetilde{h}_\eta)_k \tag{4.9}$$
$$[\mathrm{i}\langle k,\sigma\rangle \operatorname{Id} - \operatorname{ad}\Omega(\nu)]\, W_{2,k} = (\widetilde{h}_\zeta)_k \tag{4.10}$$

and, similarly, for $k = 0$

$$-\Lambda_1 = \widetilde{f}_0 \tag{4.11}$$
$$V_{2,0}\Omega(\nu) = (\widetilde{g}_\zeta)_0 \tag{4.12}$$
$$-\Omega(\nu)W_{0,0} = \widetilde{h}_0 \tag{4.13}$$
$$-\Omega(\nu)W_{1,0} = (\widetilde{h}_\eta)_0 \tag{4.14}$$
$$-\operatorname{ad}\Omega(\nu)W_{2,k} - \Omega(\Lambda_2) = (\widetilde{h}_\zeta)_0. \tag{4.15}$$

On the one hand, it is clear by the Diophantine conditions that for $k \neq 0$ none of the coefficients at the right hand sides of (4.4)-(4.9) is in the kernel, i.e. none of the eigenvalues $\mathrm{i}\langle k, \sigma\rangle$, $\mathrm{i}\langle k, \sigma\rangle \pm \lambda_j$, with λ_j eigenvalue of $\Omega(\nu)$ are zero. For $k = 0$, the equations (4.12)-(4.14) are solvable by the non-degeneracy condition BHT(i) since the right hand sides lie in $\mathcal{B}^-$. The so-called solvability condition (4.11) determines the ∂_ξ-component

$$\Lambda_1(\sigma, \nu) = -\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \widetilde{f}(\xi, 0, 0, \sigma, \nu)\, \mathrm{d}\xi$$

of N in (4.1). Turning our attention to equation (4.10), we see that it admits the solution

$$W_{2,k} = [\mathrm{i}\langle k, \sigma\rangle \,\mathrm{Id} - \mathrm{ad}\,\Omega(\nu)]^{-1} \,(\widetilde{h}_\xi)_k$$

if and only if the operator $[\mathrm{i}\langle k, \sigma\rangle \,\mathrm{Id} - \mathrm{ad}\,\Omega(\nu)]$ is invertible, which boils down to the condition

$$\mathrm{i}\langle k, \sigma\rangle \neq \lambda_j - \lambda_l$$

on the spectrum of $\mathrm{ad}\,\Omega(\nu)$, where λ_j is an eigenvalue of $\Omega(\nu)$. This inequality is the second Mel'nikov condition and again guaranteed by the Diophantine conditions. For $k = 0$ the splitting

$$\mathrm{im}(\mathrm{ad}_+(\Omega_0)) \oplus \ker(\mathrm{ad}_-(\Omega_0^T)) = \mathfrak{gl}_-(2p, \mathbb{R}), \tag{4.16}$$

detailed in the Appendix lies at the basis of solving equation (4.15). Indeed, the non-degeneracy condition BHT(ii) guarantees that we may choose the LCU for Ω. Using the Implicit Function Theorem and the fact that Ω (by construction) is an isomorphism between parameter spaces, it follows that (4.15) admits the solution

$$\Lambda_2(\sigma, \nu) = \Omega^{-1}\Big(-\pi\,\Big(\widetilde{h}_{\zeta,0} + \mathrm{ad}\,\Omega(\nu) W_{2,0}\Big)\Big), \tag{4.17}$$

where the mapping π denotes the projection of $\mathfrak{gl}_-(n, \mathbb{R})$ onto the subspace $\ker(\mathrm{ad}_-(\Omega_0^T))$ according to the splitting (4.16). Compare with [15], Lemma 8.1.

A Unfolding reversible linear matrices

Let $\Omega_0 \in \mathfrak{gl}_-(2p; \mathbb{R})$ be given; the aim of this appendix is to summarize some results from [15, 20, 23] which allow to describe a miniversal unfolding of Ω_0, and to work out the details for two particular cases.

Let $\Omega_0 = \mathcal{S}_0 + \mathcal{N}_0$ be the Jordan-Chevalley decomposition of Ω_0 into commuting semisimple and nilpotent parts. The uniqueness of this decomposition implies that both $\mathcal{S}_0$ and $\mathcal{N}_0$ belong to $\mathfrak{gl}_-(2p; \mathbb{R})$. Also

$$\ker \mathrm{ad}(\Omega_0) = \ker \mathrm{ad}(\mathcal{S}_0) \cap \ker \mathrm{ad}(\mathcal{N}_0), \tag{A.1}$$

as easily follows from the fact that $\mathcal{S}_0$ and $\mathcal{N}_0$ commute, and as shown in [15, 23] furthermore

$$\ker \mathrm{ad}(\Omega_0^T) = \ker \mathrm{ad}(\mathcal{S}_0) \cap \ker \mathrm{ad}(\mathcal{N}_0^T). \tag{A.2}$$

We know from (3.4) that $T_{\Omega_0}\mathcal{O}(\Omega_0) = \mathrm{im}\,(\mathrm{ad}_+(\Omega_0))$, while a classical result from linear algebra shows that the subspace $\ker(\mathrm{ad}_-(\Omega_0^T))$ of $\mathfrak{gl}_-(2p; \mathbb{R})$ forms a complement of the tangent space $T_{\Omega_0}\mathcal{O}(\Omega_0)$ to the orbit through Ω_0. Finally, $\ker(\mathrm{ad}_-(\Omega_0^T)) = \ker(\mathrm{ad}_-(\mathcal{S}_0)) \cap \ker(\mathrm{ad}_-(\mathcal{N}_0^T))$ by (A.2), and hence we obtain the following result.

Theorem 7 *Let $\Omega_0 \in \mathfrak{gl}_-(2p;\mathbb{R})$ be given, and let $\Omega_0 = \mathcal{S}_0 + \mathcal{N}_0$ be the Jordan-Chevalley decomposition of Ω_0. Then*

$$\Omega : \ker\left(\mathrm{ad}_-(\mathcal{S}_0)\right) \cap \ker\left(\mathrm{ad}_-(\mathcal{N}_0^T)\right) \longrightarrow \mathfrak{gl}_-(2p;\mathbb{R}), \quad A \mapsto \Omega_0 + A, \tag{A.3}$$

forms a miniversal unfolding of $\Omega_0 \in \mathfrak{gl}_-(2p;\mathbb{R})$.

The unfolding $\Omega(\mu)$ is in the centralizer of $\mathcal{S}_0$. In the present context of linear systems one calls such an unfolding a linear centralizer unfolding (LCU for short). Also note that $\Omega(\mu) - \Omega_0$ is linear in the unfolding parameters. For the convenience of the reader we now explicitly work out a linear centralizer unfolding (A.3) for three particular choices of (Ω_0, R).

A.1 Unfolding multiple non-zero normal frequencies

For our first example we assume that $\Omega_0 \in \mathfrak{gl}_-(2p;\mathbb{R})$ has a $1:1:\cdots:1$ resonance (or p-fold resonance), meaning that Ω_0 has a pair of purely imaginary eigenvalues, say $\pm\mathrm{i}$, with algebraic multiplicity p; we furthermore assume that we are in the generic situation, with geometric multiplicity 1. The subspaces $\ker(\mathcal{N}_0^j)$ $(1 \le j \le p)$ form a strictly increasing sequence of subspaces invariant under $\mathcal{S}_0$ and R, with $\dim\ker(\mathcal{N}_0^j) - \dim\ker(\mathcal{N}_0^{j-1}) = 2$. With respect to a conveniently chosen basis $\{u_1^+, u_1^-, u_2^+, u_2^-, \ldots, u_p^+, u_p^-\}$ of $\mathbb{R}^{2p}$ the linear matrices Ω_0 and R have the matrix form

$$\Omega_0 = \begin{pmatrix} \mathrm{J}_2 & \mathrm{J}_2 & \mathrm{O}_2 & \ldots & \mathrm{O}_2 \\ \mathrm{O}_2 & \mathrm{J}_2 & \mathrm{J}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathrm{O}_2 \\ \vdots & & \ddots & \ddots & \mathrm{J}_2 \\ \mathrm{O}_2 & \ldots & \ldots & \mathrm{O}_2 & \mathrm{J}_2 \end{pmatrix}, \quad R = \begin{pmatrix} R_2 & \mathrm{O}_2 & \mathrm{O}_2 & \ldots & \mathrm{O}_2 \\ \mathrm{O}_2 & R_2 & \mathrm{O}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathrm{O}_2 \\ \vdots & & \ddots & \ddots & \mathrm{O}_2 \\ \mathrm{O}_2 & \ldots & \ldots & \mathrm{O}_2 & R_2 \end{pmatrix} \tag{A.4}$$

with

$$\mathrm{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathrm{O}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.5}$$

From this one can compute an LCU of Ω_0 as follows.

Lemma 8 *Fix an $A \in \ker\left(\mathrm{ad}_-(\mathcal{S}_0)\right) \cap \ker\left(\mathrm{ad}_-(\mathcal{N}_0^T)\right)$. Then there exist constants $\mu_1, \mu_2, \ldots, \mu_p \in \mathbb{R}$ such that if we set*

$$A_j := A - \sum_{i=1}^{j} \mu_i \mathcal{S}_0^i \left(\mathcal{N}_0^T\right)^{i-1}, \qquad (1 \le j \le p),$$

then $A_j(U_{p-i}) = \{0\}$ for $1 \le j \le p$ and $0 \le i \le j-1$. In the particular case that $j = p$ we have

$$A = \sum_{i=1}^{p} \mu_i \mathcal{S}_0^i \left(\mathcal{N}_0^T\right)^{i-1}. \tag{A.6}$$

Combining (A.4) and (A.6) an LCU of Ω_0 takes the explicit form

$$\Omega(\mu) = \Omega_0 + \begin{pmatrix} \mu_1 \mathrm{J}_2 & \mathrm{O}_2 & \mathrm{O}_2 & \dots & \mathrm{O}_2 \\ \mu_2 \mathrm{J}_2 & \mu_1 \mathrm{J}_2 & \mathrm{O}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathrm{O}_2 \\ \vdots & & \ddots & \ddots & \mathrm{O}_2 \\ \mu_p \mathrm{J}_2 & \dots & \dots & \mu_2 \mathrm{J}_2 & \mu_1 \mathrm{J}_2 \end{pmatrix} \tag{A.7}$$

with unfolding parameters $\mu_1, \dots, \mu_p \in \mathbb{R}$. This construction invariably leads to the same LCU, we therefore speak from now on of *the* LCU. In case all eigenvalues of $\Omega_0 \in \mathfrak{gl}_-(2p;\mathbb{R})$ are purely imaginary, non-zero and with geometric multiplicity 1, the LCU of Ω_0 can be obtained by considering the different pairs of eigenvalues $\pm \mathrm{i}\alpha_j$, multiplying (A.7) with α_j (using for each j the appropriate dimension and a new set of parameters), and juxtaposing the obtained unfoldings as blocks along the diagonal.

A.2 Unfolding multiple eigenvalue zero

For our second and third example we assume that $\Omega_0 \in \mathfrak{gl}_-(2p;\mathbb{R})$ has 0 as an eigenvalue with geometric multiplicity 1 and algebraic multiplicity $2p$; then $\mathcal{S}_0 = 0$, $\mathcal{N}_0 = \Omega_0$, $\mathcal{N}_0^j \neq 0$ for $1 \leq j < 2p$, and $\mathcal{N}_0^{2p} = 0$. The subspaces $\ker(\mathcal{N}_0^j)$, $1 \leq j \leq 2p$ are invariant under R; they form a strictly increasing sequence, with $\dim\ker(\mathcal{N}_0^j) - \dim\ker(\mathcal{N}_0^{j-1}) = 1$. With respect to a conveniently chosen basis $\Omega_0 = \mathcal{N}_0$ is a classical nilpotent Jordan matrix with 1's above the diagonal. The matrix form of R depends on whether $\ker(\mathcal{N}_0) \subset \mathrm{Fix}(R)$, in which case R has the same matrix form as in (A.4), or $\ker(\mathcal{N}_0) \subset \mathrm{Fix}(-R)$, whence the matrix form of R equals minus the expression in (A.4).

To determine the LCU of Ω_0 we first consider some $A \in \ker(\mathrm{ad}(\mathcal{N}_0^T))$; one easily shows that A can be written as $A = \sum_{j=1}^{2p} \nu_j \left(\mathcal{N}_0^T\right)^{j-1}$, with some constants $\nu_j \in \mathbb{R}$ $(1 \leq j \leq 2p)$. Imposing the further condition that $A \in \mathfrak{gl}_-(2p;\mathbb{R})$ gives $\nu_j = 0$ for j odd; setting $\mu_j := \nu_{2j}$ for $1 \leq j \leq p$ we obtain then the following LCU

$$\Omega(\mu) = \Omega_0 + \sum_{j=1}^{p} \mu_j \left(\mathcal{N}_0^T\right)^{2j-1}, \quad \mu = (\mu_1, \mu_2, \dots, \mu_p) \in \mathbb{R}^p.$$

Hence Ω_0 has co-dimension p and the LCU is given by

$$\Omega(\mu) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ & 0 & 1 & 0 & 0 & \cdots & 0 \\ & & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ & & & & & \ddots & 1 \\ & & & & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & & & & \\ \mu_1 & 0 & & & & & \\ 0 & \mu_1 & 0 & & & & \\ \mu_2 & 0 & \mu_1 & 0 & & & \\ 0 & \mu_2 & 0 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \mu_p & \dots & 0 & \mu_2 & 0 & \mu_1 & 0 \end{pmatrix}, \tag{A.8}$$

alternating diagonals with unfolding parameters μ_j and diagonals with 0. Note that we may alternatively fix R to be of the form (A.4) and obtain the two cases by taking (A.8) and its transpose, with $\Omega_0 = \mathcal{N}_0$ having its 1's below the diagonal.

In case the condition $\dim\ker(\Omega_0) = 1$ on the geometric multiplicity of the zero eigenvalue is dropped the unfolding changes drastically and requires more parameters, i.e., has higher codimension. The same is true for our first example (non-zero normal frequencies). Further information on these cases can be found in [20].

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