Perturbation theory (dynamical systems)

Henk Broer and Heinz Hanßmann

The principle of **perturbation theory** is to study dynamical systems that are small perturbations of `simple' systems. Here simple may refer to `linear' or `integrable' or `normal form truncation', etc. In many cases general `dissipative' systems can be viewed as small perturbations of Hamiltonian systems. Focussing on Parametrized KAM Theory, persistent occurrence of quasi-periodic tori is established, both inside and outside the class of Hamiltonian systems. Typically perturbation theory explains only part of the dynamics, and in the resulting `gaps' the orderly unperturbed motion is replaced by random or chaotic motion.

Contents

Overview

The Perturbation Problem

The aim of perturbation theory is to approximate a given dynamical system by a more familiar one, regarding the former as a perturbation of the latter. The problem is to deduce dynamical properties from the `unperturbed' to the `perturbed' case. For general reading and some references see (Broer and Hanßmann 2008).

Frequently used `unperturbed' systems are

- Linear systems;
- Integrable Hamiltonian systems, compare with (Hanßmann 2007, 2008) and references therein;
- Normal form truncations, compare with (Broer 2008) and references therein;

For simplicity all systems are assumed to be `sufficiently' smooth, i.e., of class $C^\infty$ or real analytic. Moreover $\varepsilon$ is a real parameter. The `unperturbed' case corresponds to $\varepsilon = 0$ and the `perturbed' one to $\varepsilon \neq 0$ or $\varepsilon > 0.$

Examples of perturbation problems

- Hamiltonian systems with small damping (Broer and Hanßmann 2008) and references therein;
- Autonomous systems with small time-dependent terms, e.g., see (Verhulst 2008) and references therein.
- The solar system as a number of uncoupled 2-body problems, where the interaction between the planets is considered small (Arnold and Avez 1967), (Arnold 1978) and (Meyer and Hall 1992).
- Systems near equilibrium, where higher order terms are considered small, see the next section and (Broer 2008) and references therein.

Persistent properties

A central theme in Perturbation Theory is to continue equilibrium and periodic solutions to the perturbed system, applying the Implicit Function Theorem. Consider a system of differential equations

$$\dot{x} = f(x, \varepsilon), \; x \in \mathbb{R}^n, \; \varepsilon \in \mathbb{R}, \tag{1}$$

$f : \mathbb{R}^{n+1} \to \mathbb{R}^n.$ Equilibria are given by the equation $f(x, \varepsilon) = 0.$ Assuming that $f(x_0, 0) = 0$ and that

$$\mathrm{D}_x f(x_0, 0) \text{ has maximal rank}$$

the Implicit Function Theorem guarantees existence of a local arc $\varepsilon \mapsto x(\varepsilon)$ with $x(0) = x_0$ such that

$$f(x(\varepsilon), \varepsilon) \equiv 0,$$

expressing persistence of the equilibrium. Next, let the system (1) for $\epsilon = 0$ have a periodic orbit γ_0. Let Σ be a local transversal section of γ_0 and $P_0 : \Sigma \to \Sigma$ the corresponding Poincaré map. Then P_0 has a fixed point $x_0 \in \Sigma \cap \gamma_0$. By transversality, for $|\varepsilon|$ small, a local Poincaré map $P_\varepsilon : \Sigma \to \Sigma$ of (1) is well-defined. Fixed points x_ε of P_ε correspond to periodic orbits γ_ε of (1). The equation $P_\varepsilon(x(\varepsilon)) = x(\varepsilon)$ with $x(0) = x_0$ can be solved as before.

The Implicit Function Theorem guarantees the existence of critical elements like equilibrium or periodic solutions, these can be approximated by means of series expansions. Examples are power series in the perturbation parameter ε, where by the Taylor formula approximate information can be obtained. Similarly for periodic and quasi-periodic solutions Poincaré-Lindstedt series can be used, or more general asymptotic series (Siegel and Moser 1971). The algorithmic construction of such series leads to a computer algebraic approach (Rand and Armbruster 1987) and also is helpful in numerical approximations (Simó 1989).

These persistence results can also be viewed as special cases of a general statement for *normally hyperbolic invariant manifolds* see (Hirsch et al. 1977), Theorem 4.1, where a contraction on a Banach space of graphs leads to persistence. This method also yields existence of stable and unstable manifolds (Chow et al. 1982, 1994). Numerical continuation programmes exist based on versions of the Newton method.

Persistence of quasi-periodic dynamics can be established by Kolmogorov-Arnold-Moser (KAM) Theory. For details and references see Section 3 below.

In certain cases where the critical elements are not persistent, bifurcations occur. Assuming the system depends on parameters, such a critical element can be suitably deformed, which makes visible how it fits in a generic family. For details see (Broer et al. 1990), (Chow et al. 1982, 1994), (Golubitsky et al. 1985, 1988), (Guckenheimer and Holmes 1983) and (Hanßmann 2007).

General dynamics in the Hamiltonian case

Bounded dynamics in integrable Hamiltonian systems is typically quasi-periodic, and most of the resulting Lagrangian tori persist by KAM Theory. In the complement of Lagrangian KAM tori several things are in order. For three or more degrees of freedom, Lagrangian tori can not trap solutions forever in between KAM tori. Therefore the solutions can escape, which is the case in the so-called *Arnold diffusion*. Compare with (Arnold and Avez 1967) and (Broer and Sevryuk 2008) and its references.

Nearly integrable Hamiltonian systems, in terms of the perturbation size, generically exhibit *exponentially* long adiabatic stability of the action variables, see e.g. (Niederman 2008) and (Broer and Sevryuk 2008) and their references. Moreover, in small neighborhoods of Hamiltonian KAM tori one has `superexponential stickiness' of the KAM tori and adiabatic stability of the action variables, involving the so-called Nekhoroshev estimate, see (Niederman 2008) and references therein.

Examples of Chaos

Hyperbolic equilibria, periodic orbits and lower dimensional tori of integrable Hamiltonian systems are typically connected by homo- and heteroclinic orbits, wich form separatrices. Chaos often is related to the splitting of these separatrices, that in nearly integrable Hamiltonian systems is caused by generic perturbations, for discussion e.g., see (Broer and Takens 2008).

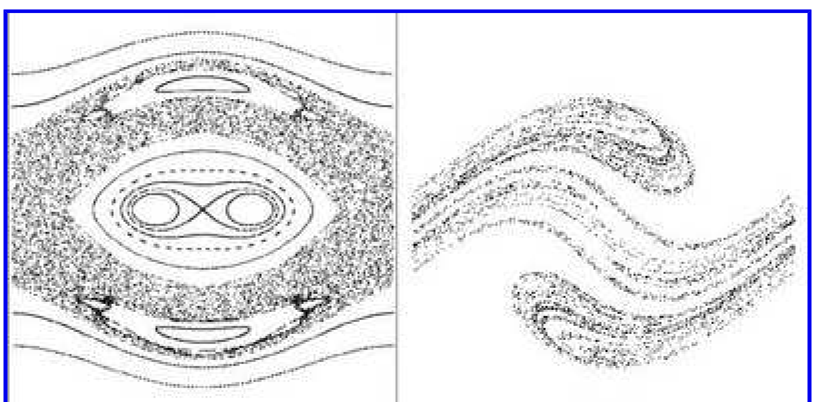

Figure 1: Chaos in the parametrically forced pendulum. Left: Poincaré map $P_{\omega,\varepsilon}$ near the $1:2$ resonance $\omega = \frac{1}{2}$ and for $\varepsilon > 0$ `not too small'. Right: A dissipative analogue.

Consider the conservative equation of motion

$$\ddot{x} + (\omega^2 + \varepsilon \cos t) \sin x = 0.$$

The corresponding (time dependent, Hamiltonian (Arnold 1978)) vector field reads

$$\dot{t} = 1, \dot{x} = y, \dot{y} = -(\omega^2 + \varepsilon \cos t) \sin x.$$

Let $P_{\omega,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ be the corresponding (area-preserving) Poincaré map. Consider the unperturbed map $P_{\omega,0}$ which is the time 2π flow of the free pendulum $\ddot{x} + \omega^2 \sin x = 0.$ For $\varepsilon \neq 0$ generically the separatrices connecting the unstable equilibrium with itself `split' and furthermore the `resonant' invariant circles filled with periodic points of the same (rational) rotation number fall apart. One method to check this is due to Melnikov, compare (Guckenheimer and Holmes 1983).

Chaos occurs in the 3-body problem as addressed, e.g., by Poincaré (Meyer and Hall 1992). A long standing open problem is to show that the clouds of points as visible in Figure 1, Left, densely fill sets of positive area, thereby leading to ergodicity (Arnold and Avez 1967). In the case with dissipation, see Figure 1, Right the occurrence is conjectured of a Hénon like strange attractor, see (Broer and Takens 2008) and references therein.

Normal Forms

For a system of autonomous differential equations (or vector field) the problem is to give a transparant local description of the dynamics by choosing appropriate coordinates. In a neighborhood of a non-equilibrium point the Flowbox Theorem asserts that, up to a change of coordinates, the vector field is constant.

Near an equilibrium point, taken as $0 \in \mathbb{R}^n$, a vector field X can be written as

$$\dot{x} = Ax + f(x), \ x \in \mathbb{R}^n, \tag{2}$$

with $A \in \mathrm{gl}(n, \mathbb{R}), \ f(0) = 0$ and $D_x f(0) = 0.$ If A is *hyperbolic* (i.e., with no purely imaginary eigenvalues), then the full (perturbed) system, near the origin, is topologically conjugated to the linear (unperturbed) system $\dot{x} = Ax.$ This is the content of the Hartman-Grobman Theorem, e.g., (Arnold 1983), (Broer et al. 1991), (Broer 2008) and (Broer and Takens 2008). So in this case the linear part is a topological normal form.

Normalizing the Taylor series

Again considering (2) the aim is to normalize the Taylor series of f at 0 step by step, for simplicity assuming that f is of class C^∞. Let $H^m(\mathbb{R}^n)$ denote the space of polynomial vector fields on $\mathbb{R}^n$, homogenous of degree m. Identifying A with the linear vector field $\dot{x} = Ax$, consider the adjoint action

$$\mathrm{ad}_m\, A : H^m(\mathbb{R}^n) \to H^m(\mathbb{R}^n), \quad Y \mapsto [A, Y],$$

where $[A, Y]$ is the Lie bracket. This map is linear. Define

$$B^m := \mathrm{im}\, \mathrm{ad}_m A,$$

the image of the map $\mathrm{ad}_m A$ in $H^m(\mathbb{R}^n)$. Then for *any* choice of complement G^m of B^m in $H^m(\mathbb{R}^n)$, in the sense that

$$B^m \oplus G^m = H^m(\mathbb{R}^n),$$

the corresponding notion of *normal form* requires that for $m \geq 2$ the homogeneous part of degree m is in G^m.

Using induction on m it turns out that for any $N \geq 2$ a real analytic change of variables $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ exists, with $\Phi(0) = 0$, such that

$$\Phi_* X(y) = Ay + g_2(y) + \cdots + g_N(y) + O(|y|^{N+1}), \tag{3}$$

with $g_m \in G^m$, for all $m = 2, 3, \ldots, N$. This result goes back to Poincaré, and has been further elaborated by Birkhoff, Sternberg and Takens, compare with (Arnold 1983), (Guckenheimer and Holmes 1983), (Broer et al. 1991) and (Broer 2008).

- If $A \in \mathrm{gl}(n, \mathbb{R})$ is semisimple, then also ad_m is, and a natural choice is $G^m = \ker \mathrm{ad}_m A$. In this case the normalized vector field terms in G^m are equivariant with respect to the linear flow e^{tA}.
- In important cases this implies a *toroidal* symmetry on the normalized truncation. Here the eigenvalues of A are purely imaginary and a finite number ($N + 1$) of resonances has to be excluded.
- In the more general case where $A = A_s + A_n$ is the Jordan Chevalley canonical splitting in semisimple and nilpotent, :$B^m + \ker \mathrm{ad}_m A_s = H^m(\mathbb{R}^n)$ and a smaller choice $G^m \subset \ker \mathrm{ad}_m As$ is suitable. In that case still equivariance with respect to e^{tA_s} holds true.
- The actual computation of the normal form terms needs to solve linear (homological) equations of the form :$\mathrm{ad}_m A(Y_m) - f_m \in G^m$, for $Y_m \in H^m(\mathbb{R}^n)$, where $f_m \in H^m(\mathbb{R}^n)$ is given by the higher order terms of (2). There exist several powerful algorithms for this, particularly also developed for the nilpotent case when $A_s = 0$, compare with (Broer et al. 2003), (Chow et al. 1994), (Sanders et al. 2007) and (Takens and Vanderbauwhede 2008).

After a couple of normalizing steps, the Perturbation Theory format (2) has been replaced by

(3), where the truncated, normalized part serves as the unperturbed system (Chow et al. 1994), (Takens and Vanderbauwhede 2008).

The local approach just described also works near periodic solutions and quasi-periodic tori, as well as near fixed and periodic points of diffeomorphisms, see (Broer 2008) and references. Although the normalized series, or the normalizing transformations, are typically divergent, in certain cases convergence can be established (Walcher 2008).

Preservation of structure

The results of Section 2.1 also hold when a given structure has to be preserved, such as a symplectic or volume form, a symmetry group (both equivariant and reversible), or when external parameters are present in the systems (also combinations are allowed). A natural language is that of Lie-subalgebras of the general Lie-algebra of vector fields and the corresponding Lie-subgroup of the general Lie-group of diffeomorphisms (Broer et al. 1991), (Broer 2008). This unification, among other things, includes the Birkhoff Normal Form Theory of Hamiltonian systems. Indeed, if for a given symplectic structure σ the Hamiltonian vector field corresponding to a Hamiltonian function H is indicated by $X_H,$ i.e., with $dH = \sigma(X_H, -),$ then, since

$$X_{\{H,G\}} = [X_H, X_G],$$

the map $H \mapsto X_H$ is a morphism of Lie-algebras. Here $\{H, G\}$ denotes the Poisson bracket of H and G.

There exist many applications of the ensuing perturbation problem, e.g, see (Broer et al. 2003), (Broer and Hanßmann 2008), (Hanßmann 2007) and references. In many cases also Singularity Theory is used.

Parametrized KAM Theory

KAM Theory concerns the typical occurrence of quasi-periodic tori in dynamical systems, i.e., persistent under sufficiently small perturbations. In all cases, in the product of phase space and parameter space, the quasi-periodic tori are Whitney-smoothly parametrized over a nowhere dense set of positive measure (involving a Cantor set), (Broer and Hanßmann 2008), (Broer et al. 1996), (Broer et al. 1990), (Broer and Sevryuk 2008), (Chierchia 2008), (Hasselblatt and Katok 2006), (Pöschel1982) and (Zehnder 1975, 1976).

KAM Theory started with Lagrangian tori in nearly-integrable Hamiltonian systems, but the theory allows for a Lie-algebra approach, which generalizes to equivariant or reversible systems. This also holds for the class of general smooth systems, called `dissipative'. It turns out that in many cases parameters are needed for persistence of the tori.

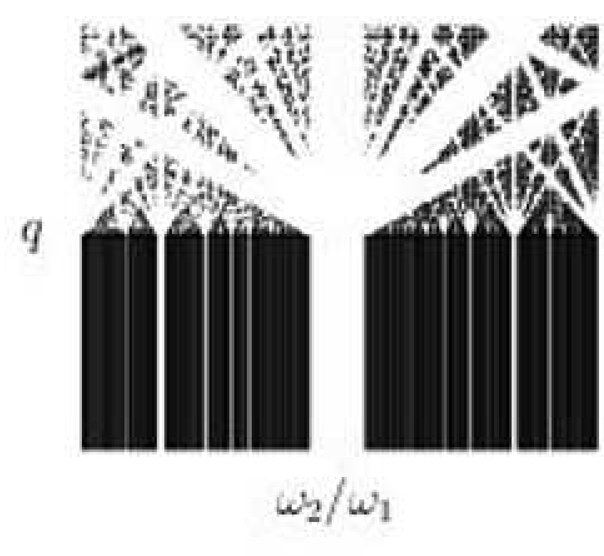

Figure 2: Sketch of the Cantorized Fold, as this occurs in many bifurcational settings, for instance in the quasi-periodic Hopf bifurcation from 2- to 3-tori. This is where quasi-periodicity and chaos are close together. This plays a role in the Ruelle-Takens scenario for the onset of turbulence.

Families of quasi-periodic attractors

In the dissipative setting consider parametrized systems with normally hyperbolic invariant n -tori. Following (Hirsch et al. 1977) this system can be restricted to the invariant torus, i.e., to the n -torus $\mathbb{T}^n = \{x \pmod{2\pi\mathbb{Z}^n}\}$, which then is the phase space. Here consider

$$\dot{x} = \omega(\mu) + \varepsilon f(x, \mu, \varepsilon)\dot{\mu} = 0, \quad (4)$$

where $\mu \in \mathbb{R}^n$ is a multi-parameter. The results of the classical KAM Theorem (Pöschel 1982) largely carry over to (4).

For $\varepsilon = 0$ (4) is `integrable' (Broer et al. 1990) and an open subset of $\mathbb{T}^n \times \mathbb{R}^n$ is foliated by invariant n -tori. The question is in how far the dynamics on the resulting invariant tori is quasi-periodic. The answer is analoguous to the Hamiltonian case. Instead of the Kolmogorov non-degeneracy condition the frequency map $\mu \mapsto \omega(\mu)$ needs to be a (local) diffeomorphism. As a consequence (4)$_{\mu,\varepsilon}$ is Whitney-smoothly conjugated to (4)$_{\mu,0}$, provided that the map ω is co-restricted to the Diophantine set $\mathbb{R}^n_{\tau,\gamma}$, defined by

$$\mathbb{R}^n_{\tau,\gamma} = \{\omega \in \mathbb{R}^n \mid |\langle k, \omega\rangle| \geq \gamma |k|^{-1} \text{ for all } k \in \mathbb{Z} \setminus \{0\}\}. \tag{5}$$

Here $\langle k, \omega \rangle$ is the standard inner product and $|k| = \sum_j |k_j|.$ For a proof see (Broer et al. 1996).

- This result holds for C^∞-systems, but also in C^ℓ with ℓ sufficiently large see the above references. The formulation is in terms of (structural) stability restricted to a suitable union of Diophantine quasi-periodic tori, for the occasion baptized as *quasi-periodic stability*.
- Dissipative KAM Theory gives rise to families of quasi-periodic attractors that occur typically. This is of importance in center manifold reductions of infinite dimensional dynamics as, e.g., in fluid mechanics (Broer and Hanßmann 2008) and references.
- In cases where the system is degenerate, for instance because there is `lack of parameters', a path formalism can be invoked, where the parameter `path' is required to be a generic subfamily of the Diophantine set $\mathbb{R}^n_{\tau,\gamma}.$ This amounts to the Rüssmann non-degeneracy, that still gives positive measure of quasi-periodicity in the parameter space, compare with (Broer et al. 1996 2007) and references.

Lower dimensional tori

The above approach extends to cases where the dynamics transversal to the tori is taken into account. For the history (that starts with Moser in the 1960s) and for details see (Broer et al. 1996, 2008) as well as (Broer and Hanßmann 2008), including all references.

Consider the phase space $\mathbb{T}^n \times \mathbb{R}^m = \{x \,(\mathrm{mod}\ 2\pi\mathbb{Z}^n), y\}$ and a parameter space $\{\mu\} = P \subset \mathbb{R}^s.$ For $\mu = 0 \in P$ the smooth `integrable' vector field $X = X(x, y, \mu)$

$$\dot{x} = \omega(\mu) + f(y, \mu), \dot{y} = \Omega(\mu)\, y + g(y, \mu), \dot{\mu} = 0, \tag{6}$$

has $\mathbb{T}^n \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^m$ as an invariant n-torus, with $f(y,0) = O(|y|)$ and $g(y,0) = O(|y|^2),$ so the invariant torus is assumed to be reduced to Floquet form. Again the question is to what extent this torus and its dynamics is persistent under small `near-integrable' perturbation to a system $\tilde{X} = \tilde{X}(x, y, \mu)$.

- Broer-Huitema-Takens (BHT) non-degeneracy (the present analogue of Kolmogorov non-degeneracy) requires that the product map $\omega \times \Omega : P \to \mathbb{R}^n \times \mathrm{gl}(m, \mathbb{R})$ is a versal unfolding of $(\omega(0), \Omega(0))$ (Arnold 1983), (Broer et al. 1990), (Broer and Sevryuk 2008) and references. In the case of simple eigenvalues normal form unfoldings exist where the eigenvalues of $\Omega(\mu)$ take the role of parameters.
- The present Diophantine conditions generalize (5), also including the normal frequencies of $\Omega(\mu)$, i.e., the imaginary parts $\beta_1, \ldots, \beta_N$ of its non-real eigenvalues as follows: Given $\tau > n-1$ and $\gamma > 0$, for all $k \in \mathbb{Z}^n \setminus \{0\}$ and all $\ell \in \mathbb{Z}^N$ with $|\ell| \leq 2$ that

$$|\langle k, \omega \rangle + \langle \ell, \beta \rangle| \geq \gamma |k|^{-\tau}. \qquad (7)$$

As a subset of P, this again defines a nowhere dense set of positive measure.

The ensuing Parametrized KAM Theory states quasi-periodic stability of the n-tori under consideration, thereby yielding typical examples where quasi-periodicity has positive measure in parameter space. Moreover, the normal linear behaviour of the n-tori is preserved by Whitney smooth conjugations. This is of importance for quasi-periodic bifurcations.

- The above set-up allows for a stucture preserving formulation as mentioned earlier, thereby including the Hamiltonian and volume preserving case, as well as equivariant and reversible cases. Compare with the discussion in Section 2.2.
- Parametrized KAM Theory *a priori* needs many parameters. Often the parameters are `distinguished' in the sense that they are given by action variables, etc. This holds, e.g., for isotropic tori in nearly integrable Hamiltonian systems. Also compare with the discussion on Rüssmann non-degeneracy at the end of Section 3.1.
- Parametrized KAM Theory leads to quasi-periodic versions of the bifurcation theory for equilibria and periodic solutions. In the dissipative setting this includes quasi-periodic saddle-node and period doubling bifurcations, as well as the quasi-periodic Hopf bifurcation. In the quasi-periodic Hopf case invariant $(n+1)$-tori branch off from invariant n-tori when the latter lose normal hyperbolicity. The bifurcation is even more involved than the Hopf-Neimark-Sacker bifurcation where an invariant 2-torus bifurcates from a periodic solution (Broer et al. 1990, 1996). Similar quasi-periodic bifurcation scenarios exist in the Hamiltonian and the reversible case, where for instance quasi-periodic versions exist of the Hopf bifurcation, see (Broer and Hanßmann 2008), (Broer et al. 1990), (Broer and Sevryuk 2008), (Hanßmann 2007) and references.
- Quasi-periodic bifurcation theory concerns bifurcations to invariant tori in nearly-integrable systems, e.g., when the tori lose their normal hyperbolicity or when certain (strong) resonances occur. In that case the dense set of resonances, responsable

for the small divisors, leads to a `Cantorisation' of the classical bifurcation geometries obtained from Singularity Theory (Broer and Hanßmann 2008), (Broer et al. 1990) (Broer and Sevryuk 2008), (Hanßmann 2007). Compare with the above figure 2.

References

H.W. Broer, Normal forms in perturbation theory. In: R. Meyers (ed.), *Encyclopaedia of Complexity & System Science*. Springer-Verlag 2008 (to appear).

H.W. Broer and H. Hanßmann, Hamiltonian perturbation theory (and transition to chaos). In: R. Meyers (ed.), *Encyclopaedia of Complexity & System Science*. Springer-Verlag 2008 (to appear).

L. Chierchia, Kolmogorov-Arnold-Moser KAM Theory. In: R. Meyers (ed.), *Encyclopaedia of Complexity & System Science*. Springer-Verlag, 2008 (to appear).

H. Hanßmann, Dynamics of Hamiltonian systems. In: R. Meyers (ed.), *Encyclopaedia of Complexity & System Science*. Springer-Verlag, 2008 (to appear).

L. Niederman, Nekhoroshev Theory. In: R. Meyers (ed.), *Encyclopaedia of Complexity & System Science*. Springer-Verlag, 2008 (to appear).

F. Verhulst, Perturbation analysis of parametric resonance. In: R. Meyers (ed.), *Encyclopaedia of Complexity & System Science*. Springer-Verlag, 2008 (to appear).

S. Walcher, Convergence of Perturbative Expansions. In: R. Meyers (ed.), *Encyclopaedia of Complexity & System Science*. Springer-Verlag, 2008 (to appear).

H.W. Broer and M.B. Sevryuk, KAM Theory: quasi-periodicity in dynamical systems. In: H.W. Broer, B. Hasselblatt and F. Takens (eds.), *Handbook of Dynamical Systems* Vol. 3, North-Holland, 2008 (to appear).

B. Hasselblatt and A. Katok (eds.), *Handbook of Dynamical Systems* Vol. 1B, North-Holland, 2006.

F. Takens and A. Vanderbauwhede, Local invariant manifolds and normal forms. In: H.W. Broer, B. Hasselblatt and F. Takens (eds.), *Handbook of Dynamical Systems* Vol. 3, North-Holland, 2008 (to appear).

V.I. Arnold, *Mathematical Methods of Classical Mechanics*, GTM **60** Springer-Verlag, New York, 1978.

V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer-Verlag, 1983.

V.I. Arnold and A. Avez, *Problèmes Ergodiques de la Mécanique classique*, Gauthier-Villars, 1967; *Ergodic problems of classical mechanics*, Benjamin, 1968.

H.W. Broer, F. Dumortier, S.J. van Strien and F. Takens, Structures in dynamics, finite dimensional deterministic studies. In E.M. de Jager and E.W.C. van Groesen (eds.) *Studies in Mathematical Physics II* North-Holland, 1991.

H.W. Broer, I. Hoveijn, G.A. Lunter and G. Vegter, *Bifurcations in Hamiltonian Systems: Computing Singularities by Gröbner Bases*, Lecture Notes in Mathematics **1806**, Springer-Verlag, 2003.

H.W. Broer, G.B. Huitema and M.B. Sevryuk, *Quasi-Periodic Motions in Families of Dynamical Systems: Order amidst Chaos*, Lecture Notes in Mathematics **1645**, Springer-Verlag, 1996.

H.W. Broer, G.B. Huitema, F. Takens and B.L.J. Braaksma, Unfoldings and bifurcations of quasi-periodic tori, *Memoir AMS*, **421**, Amer. Math. Soc., Providence, R.I. 1990.

H.W. Broer and F. Takens, *Dynamical Systems and Chaos,* Epsilon Uitgaven, 2008 (to appear).

S.-N. Chow and J.K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, 1982.

S-N. Chow, C. Li and D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields.* Cambridge University Press, 1994.

M. Golubitsky and D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory, Volume I*, Applied Mathematical Sciences **51**, Springer, 1985.

M. Golubitsky, I.N. Stewart and D.G. Schaeffer, *Singularities and Groups in Bifurcation Theory, Volume II*, Applied Mathematical Sciences **69**, Springer, 1988.

J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, 1983.

H. Hanßmann, *Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems – Results and Examples*, Lecture Notes in Mathematics **1893**, Springer-Verlag, 2007.

M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant Manifolds*, Lecture Notes in Mathematics **583**, Springer-Verlag, 1977.

K.R. Meyer and G.R. Hall, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, Applied Mathematical Sciences **90**, Springer, 1992.

J. Pöschel, Integrability of Hamiltonian systems on Cantor sets. *Comm. Pure Appl. Math.* **35**(5) (1982), 653-696.

R.H. Rand and D. Armbruster, *Perturbation methods, bifurcation theory and computer algebra*, Springer-Verlag 1987.

J.A. Sanders, F. Verhulst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*. Revised 2nd Edition, Appl Math Sciences **59**, Springer-Verlag 2007.

C.L. Siegel and J.K. Moser, *Lectures on Celestial Mechanics*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen **187**, Springer-Verlag 1971.

C. Simó, On the analytical and numerical approximation of invariant manifolds. In: D. Benest and C. Froeschlé (eds.), Les Méthodes Modernes de la Mécanique Celeste, 285-329, Goutelas, 1989.

E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems, I and II, *Comm. Pure Appl. Math.*, **28**(1) (1975), 91-140; **29**(1) (1976), 49-111.

Recommended Reading

H.W. Broer, B. Hasselblatt and F. Takens, *Handbook of Dynamical Systems* Vol. 3, North-Holland, 2008 (to appear).

R.V. Gamkredlidze, *Encyclopaedia of Mathematical Sciences*, volumes on Dynamical Systems, in particular Vol. 3 by V.I. Arnold, V.V. Kozlov and A.I. Neishtadt, VINITI & Springer, 1985-2006.

R. Meyers, *Encyclopaedia of Complexity & System Science*, section on Perturbation Theory (ed. G. Gaeta), Springer-Verlag 2008 (to appear).