On the destruction of resonant Lagrangean tori in Hamiltonian systems

Henk W. Broer
Instituut voor Wiskunde en Informatica, Rijksuniversiteit Groningen
Nijenborgh 9, 9747 AG Groningen, The Netherlands

Heinz Hanßmann
Mathematisch Instituut, Universiteit Utrecht
Postbus 80010, 3508 TA Utrecht, The Netherlands

Jiangong You
Department of Mathematics, Nanjing University
Nanjing 210093, P. R. China

31 october 2012

Abstract

Starting from Poincaré's fundamental problem of dynamics, we consider perturbations of integrable Hamiltonian systems in the neighbourhood of resonant Lagrangean (i.e. maximal) invariant tori with a single (internal) resonance. Applying KAM Theory and Singularity Theory we investigate how such a torus disintegrates when the action variables vary in the resonant surface. For open subsets of this surface the resulting lower dimensional tori are either hyperbolic or elliptic. For a better understanding of the dynamics, both qualitatively and quantitatively, we also investigate the singular tori and the way in which they are being unfolded by the action variables. In fact, if N is the number of degrees of freedom, singularities up to co-dimension $N-1$ cannot be avoided. In the case of Kolmogorov non-degeneracy the singular tori are parabolic, while under the weaker non-degeneracy condition of Rüssmann the lower dimensional tori may also undergo *e.g.* umbilical bifurcations. We emphasise that this application of Singularity Theory only uses internal (or distinguished) parameters and no external ones.

1 Introduction

Classical perturbation theory largely concerns the continuation of quasi-periodic motions as these occur in integrable Hamiltonian systems for small non-integrable perturbations. The classical perturbation series here diverge on a resonance subset that densely fills the phase space, leading to the notorious small denominators even when avoiding this dense set. This paper deals with the dynamics within such resonance gaps.

Background. We briefly summarize that Kolmogorov–Arnol'd–Moser (KAM) Theory [2] establishes the persistence of quasi-periodic motions that densely fill Lagrangean tori, meaning that the dimension equals the number N of degrees of freedom. If $\omega \in \mathbb{R}^N$ denotes the frequency vector, the resonances alluded to above are given by

$$\langle k, \omega \rangle \; = \; 0 \;, \quad 0 \neq k \in \mathbb{Z}^N \;. \tag{1}$$

KAM Theory excludes such resonances by imposing strongly non-resonant, Diophantine conditions

$$|\, \langle k, \omega \rangle \,| \;\; \geq \;\; \frac{\gamma}{|k|^\tau} \quad \text{for all } k \in \mathbb{Z}^N \backslash \{0\} \tag{2}$$

on the frequency vectors, where $\gamma > 0$, $\tau > N-1$, which guarantee the persistence of many Lagrangean tori in the sense of measure theory. We recall [25] that the persistent N–tori are smoothly parametrised over the nowhere dense union of closed half lines defined by (2), in the sense of Whitney; colloquially we speak of a *Cantor family of half lines*.

Aim of the paper. We investigate what happens to the Lagrangean tori of the unperturbed system for which the frequency vector is resonant, *i.e.* satisfying (1) for a single fixed nonzero $k \in \mathbb{Z}^N$ (and its integer multiples), so which is contained in a gap of the 'Cantor set' defined by (2).

Starting point is the real analytic perturbed N–degree-of-freedom Hamiltonian

$$H_\varepsilon(\varphi, I) \;\; = \;\; H_0(I) \; + \; \varepsilon H_1(\varphi, I; \varepsilon) \tag{3}$$

defined on $\mathbb{T}^N \times \mathbb{R}^N$ with perturbation parameter $0 < \varepsilon \ll 1$. We concentrate on a single resonance, whence a normal form approximation can be reduced to one degree of freedom. See below for details. The reduced system is defined on the cylinder $\mathbb{T} \times \mathbb{R}$ with Hamiltonian of the form

$$\mathcal{H}(p, q; \mu) \;\;, \quad (q, p) \in \mathbb{T} \times \mathbb{R}, \; \mu \in \mathbb{R}^{N-1} \;\; ; \tag{4}$$

here $\mu = \mu(I)$ is a (distinguished) parameter varying along the resonance surface

$$\langle k, \omega(I) \rangle \; = \; 0 \;, \qquad \text{where} \qquad \omega(I) \; := \; DH_0(I) \;.$$

We will show that (4) is a general family of Hamiltonian functions and our interest is the study of the dynamics they generate. As in all one-degree-of-freedom systems, this dynamics is largely determined by the configuration of level sets. We recall [2] that the closed curves correspond to librational Lagrangean N–tori, while the critical points correspond to invariant $(N-1)$–tori. The latter govern the global geometry of the dynamics and in the $(N-1)$–parameter family (4) of functions their behaviour is determined by Singularity Theory [27, 3].

Within such an $(N-1)$–parameter family of functions one may encounter singularities up to co-dimension $N-1$ in a persistent way. Singularity Theory provides us with versal unfoldings of these. These unfoldings also will appear persistently in our general family (4).

The compact-open topology. We like to note that for families of planar functions persistence in the sense of structural stability is a generic property. To formulate this property in a precise way one needs a topology on the space of Hamiltonian functions in one degree of freedom. In the present real analytic setting we use the compact-open topology on holomorphic extensions detailed in [11].

The real-analytic compact-open topology on holomorphic extensions fits with local uniform convergence of the corresponding Hamiltonian functions. We recall from [11] that this compact-open topology has the Baire property, which means that countable intersections of dense-open sets are still dense. Moreover, the compact-open topology is stronger than the Whitney C^k topologies used in [23, 22]. From the latter it immediately follows that any C^k open property also is open in the compact-open sense. The same holds for denseness, as long as we restrict to properties defined in terms of transversality. Therefore a real analytic unfolding that is versal in the C^k sense also is versal with respect to the compact-open topology. In particular the differentiable Singularity Theory also applies for the real analytic case. In fact, Weierstraß originally developed the theory for the analytic case.

Co-dimensions. We recall that for equilibria to be either hyperbolic or elliptic, the corresponding singularity A_1 has co-dimension 0, which means that in one degree of freedom it has open occurrence in the space of families of Hamiltonian functions. However, the singularities of higher co-dimension determine the geometric organization of those of lower co-dimension, therefore in particular the ones corresponding to hyperbolic and elliptic equilibria. This organization reflects both qualitative and quantitative aspects.

Example 1.1 [**A quasi-periodic centre-saddle bifurcation**] As an example[1] in the space of $(N-1)$–parameter families in one degree of freedom we consider

$$\mathcal{H}(p,q;\mu) \;=\; \frac{1}{2}p^2 \;+\; \frac{1}{6}q^3 \;+\; \lambda(\mu)q \tag{5}$$

with $\lambda(0) = 0$ and $D_\mu\lambda(0) \neq 0$. We remark that this family usually is called *fold*, a versal unfolding of the singularity A_2, see [27, 3]. Clearly (5) is a family of one-degree-of-freedom systems for which the equilibrium $(p,q) = (0,0)$ at $\mu = 0$ is parabolic (and hence neither elliptic nor hyperbolic), and for any (real analytic) small perturbation there is a parameter value $\mu = \mu_0 \approx 0$ for which there is a parabolic equilibrium at a certain point $(p,q) = (p_0,q_0) \approx (0,0)$. Thus, there is a full neighbourhood of the $(N-1)$–parameter family defined by (5) in the compact-open topology such that a parabolic equilibrium occurs nearby. Hence it is impossible for the set of all families of one-degree-of-freedom systems that only have elliptic and/or hyperbolic equilibria to contain a countable intersection of open and dense sets, *i.e.* to be generic.

In the reconstruction to the setting of N degrees of freedom, the equilibria correspond to invariant $(N-1)$–tori, where the normal behaviour is inherited, and the closed level curves

[1] Any non-degenerate example of a co-dimension 1 bifurcation can be reduced to this case. Here we exclude a setting with symmetry or other structural restrictions [20].

correspond to librational Lagrangean N–tori. Addition of the non-symmetric higher order terms presents us with a new perturbation problem. This can be solved by KAM Theory [18]. In this way the entire bifurcation scenario becomes *Cantorised* as explained below, for a detailed example also see Section 2.

An interesting aspect is that in a fixed energy level we have the same theory, with one parameter less. For $N \geq 3$ we thus have in each energy level elliptic, hyperbolic and parabolic tori in a center-saddle bifurcation. In both cases a quantitative consequence of this is that for small perturbations the asympotic distance of the two families of tori, as they approach the parabolic equilibrium, is of order $\sqrt{-\lambda(\mu)}$ as $\mu \to 0$.

Observe that for $N = 2$ the lower dimensional tori really are closed orbits. In this case the energy is the only parameter and no bifurcation takes place inside an energy level. For $N \geq 3$ the bifurcating tori have dimension larger than or equal to 2 and the bifurcation can take place within an energy level. $\triangle$

Unfoldings. The fold example (5) is a special case of the family of cuspoids that unfold the co-rank 1 singularities A_{k+1}, $k \in \mathbb{N}$, see [27, 3] for more details. The corresponding KAM perturbation problem has been solved in [5]. In that paper (trans)-versality conditions as dictated by Singularity Theory [27, 3] are used to develop a normal form of Hamiltonians in the neighbourhood of a parabolic torus. Solving the ensuing small divisor problems it is proven that the bifurcation scenario persists in a Cantorised way. See Example 2.4 below for a more detailed description. In this way all possible quasi-periodic bifurcations of normally parabolic tori can be retrieved in resonance gaps, taking N sufficiently high.

An aspect that one has to keep in mind in the present case of $(N-1)$–parameter families (4) defined by the normal form of (3) near a single resonance, is that part of the terms in $\mathcal{H}$, say the p–terms, come from H_0 and the remaining terms come from the perturbation H_1. The p^2–term in (5) occurs in the case where the integrable part H_0 satisfies the Kolmogorov condition

$$\det D^2 H_0(I) \quad \neq \quad 0 \ , \tag{6}$$

which is valid on an open and dense set in parameter space, see [2]. For these values of I the above theory of parabolic tori applies, see Section 2 for more details. In particular, one can read off from the degeneracy of the minumum and the maximum of a certain potential V_μ on $\mathbb{T}$ a lower bound for the number of families of invariant $(N-1)$–tori.

Note that the Kolmogorov non-degeneracy condition (6) fails on a co-dimension 1 subset. Here one may still expect a weaker form of non-degeneracy to hold true, expressed by the Rüssmann condition, cf. [8, 26, 29]. In this way also more general singularities can be incorporated, giving rise to all possible applications of Singularity Theory as in [20]. More degenerate singularities may *e.g.* lead to umbilical torus bifurcations, for an example see Section 3; in Section 4 we discuss further possibilities.

We emphasise that we are not to impose genericity conditions on occurring equilibria, but only on the initial Hamiltonian H, see (3). Our genericity assumption on H_0 amounts to Rüssmann non-degeneracy. The remaining genericity assumptions on H are obtained via H_1. These genericity conditions amount to versality of the occurring unfolding, thereby excluding too pathological examples. For more details see below.

On Cantorisation. The Diophantine condition (2) defines a Cantor family of closed half lines, parametrised over a Cantor set of positive measure [9]. This Cantor family in turn parametrises a Cantor bundle of integrable quasi-periodic invariant tori, in a Whitney smooth way. KAM Theory implies that in case of Kolmogorov non-degeneracy, under small perturbations this Cantor bundle is distorted by a near-identity Whitney smooth conjugation. Where the integrable invariant tori foliate a submanifold or a semi-algebraic set organized by Singularity Theory, we colloquially say that the Diophantine condition (2) 'Cantorises' this geometry, with the same terminology for nearly integrable systems. We note that the property of having positive (Hausdorff) measure is preserved by diffeomorphisms. In the sequel we shall also meet Cantor bundles of tori parametrised over (real) Cantor sets. In all cases these bundles can be distinguished by their Hausdorff dimension.

Related work. To our knowledge, invariant tori reconstructed from possibly degenerate equilibria have only been addressed for single resonances, not for multiple resonances.

Cheng [13] considers convex H_0, such that in the reduced system (4) the maximum and minimum of the q–dependent part differ. The invariant $(N-1)$–tori corresponding to non-degenerate maxima are hyperbolic tori, but in [13] a degenerate maximum is *not* excluded and the resulting tori are called of 'hyperbolic type'. It is established that the system (3) has a Cantor family of 'hyperbolic type' invariant $(N-1)$–tori. In [14] the minima of the q–dependent part of (4) are treated, restricting to non-degenerate minima (*i.e.* elliptic tori). This yields a Cantor family of elliptic invariant $(N-1)$–tori.

The approach by Gallavotti, Gentile and Giuliani [17] considers the perturbation parameter ε also as a bifurcation parameter. Although they do consider degenerate singularities, the results only concern Cantor families of elliptic and hyperbolic invariant $(N-1)$–tori. Careful considerations yield expansions of the ε–family of $(N-1)$–tori in *e.g.* $\sqrt{\varepsilon}$, and quantitative asymptotic information. We note that these results can be retrieved as a direct consequence of our approach. However, the fate of degenerate tori is not explained in [17] and *a fortiori* a complete bifurcation scenario is not discussed.

As soon as one makes the assumption that all equilibria of the reduced system are non-degenerate one obtains elliptic and hyperbolic lower dimensional tori also for multiple resonances, see [15, 12, 28, 10] and references therein. We like to emphasise that this assumption is not generic for reduced systems parametrised by the conjugate actions; one then also has to deal with degenerate equilibria and the corresponding bifurcation theory. A starting point for the development of such a persistence result is formed by [7, 4], where multiple resonances are taken into account. For m–fold resonances corank–m–singularities may occur already under the Kolmogorov condition, and under the Rüssmann condition this may further rise to corank $2m$.

2 Kolmogorov Hamiltonians

To explain the kind of results that can be obtained for the perturbation (3) of an integrable Hamiltonian $H_0 = H_0(I)$ on $\mathbb{T}^N \times \mathbb{R}^N$ we first restrict to an open subset $U \subset \mathbb{R}^N$ where the Kolmogorov non-degeneracy condition (6) is valid for every $I \in U$. By classical KAM Theory [25, 2, 8] it then follows that most Lagrangean tori $\mathbb{T}^N \times \{I\}$, $I \in U$ satisfy the Diophantine conditions (2) and persist. For Lagrangean tori with resonant frequency vector that have a single resonance we have the following result.

Theorem 2.1 *[**Resonant dynamics**] Consider the perturbed Hamiltonian* (3) *on* $\mathbb{T}^N \times U$ *satisfying the Kolmogorov non-degeneracy condition* (6) *for all* $I \in U$*. Then for sufficiently small perturbations* H_1*, satisfying suitable genericity conditions, a Lagrangean torus of the unperturbed system with a single resonance* $\langle k, DH_0(I)\rangle = 0$, $k \in \mathbb{Z}^N \backslash \{0\}$ *leads in the perturbed system* (3) *to Cantor families of hyperbolic, elliptic and possibly also parabolic tori. The distribution of these tori is determined by the way in which the genericity conditions on* H_1 *are fulfilled.*

The proof in particular reveals the nature of the genericity conditions, made precise in Lemma 2.2 and the paragraph preceeding it.

Proof. The equation $\langle k, DH_0(I)\rangle = 0$ determines a local hypersurface $\mathbb{Y} \subset U$. For $I \in \mathbb{Y}$ the unperturbed Lagrangean torus $\mathbb{T}^N \times \{I\}$ is foliated into invariant tori of dimension $N - 1$. Let us put

$$n = N - 1$$

and on U choose local co-ordinates $\varphi = (x, q)$ with values in $\mathbb{T}^n \times \mathbb{T}$ and $I = (y, p)$ with values in $\mathbb{R}^n \times \mathbb{R}$, where y parametrises the surface $\mathbb{Y}$. In these local co-ordinates the single resonance reads

$$\frac{\partial}{\partial p} H_0(y, 0) = 0 \quad \text{for all } y \in \mathbb{Y} , \tag{7}$$

cf. [13, 14, 28, 10]. The remaining frequencies form the vector $\omega = D_y H_0(y, 0)$ which is non-resonant at a single resonance $(y, p) = (y^*, 0)$. Treating ω as an external parameter, we expect persistence results only for Diophantine ω, where gaps in the resulting 'Cantor set' correspond to multiple resonances. Following [25, 9], we localize to $\hat{y}$ (ε–close to 0) writing $y = y^* + \hat{y}$, thereby shrinking the neighbourhood U if necessary. Moreover we restrict to the lowest order terms

$$H_0(\hat{y}, p; y^*, \omega) = \langle \omega, \hat{y}\rangle + \frac{a(y^*)}{2} p^2 .$$

From (6) together with (7) we infer

$$a(y^*) \neq 0 \quad \text{for all } y^* \in \mathbb{Y} ,$$

whence we find a lower bound of the function $|a|$ by shrinking the co-ordinate domain $U \supset \mathbb{Y}$ a bit if necessary.

We now apply a familiar method to replace the system (3) by a family of one-degree-of-freedom systems, cf. [2, 13, 28, 10]. Starting point is a normalizing transformation that turns the perturbed Hamiltonian into

$$H_\varepsilon(x, \hat{y}, p, q; y^*, \omega) \;=\; H_0(\hat{y}, p; y^*, \omega) \;+\; \varepsilon \bar{H}_1(\hat{y}, p, q; y^*, \omega) \;+\; \mathcal{O}(\varepsilon^2)$$

where $\bar{H}_1$ is the $\mathbb{T}^n$–average along x of H_1 at $\varepsilon = 0$. In the expansion

$$\begin{aligned}\bar{H}_1 \;&=\; \eta(\hat{y}; y^*, \omega) \;+\; \alpha(\hat{y}; y^*, \omega)p \;+\; \beta(\hat{y}; y^*, \omega)q \\ &\quad+\; \frac{A(y^*)}{2}p^2 \;+\; \frac{B(y^*)}{2}q^2 \;+\; C(y^*)pq \;+\; \ldots\end{aligned}$$

we may have $A(y^*) \equiv 0$, but more importantly $|a(y^*) + \varepsilon A(y^*)|$ is still bounded from below on $\mathbb{Y}$. Re-parametrising $\omega \mapsto \omega + \varepsilon D_{\hat{y}}\eta(\hat{y}; y^*, \omega)$, maintaining the same symbol ω for the frequency vector, the expansion of H_ε still starts with $\langle \omega, \hat{y} \rangle$. By means of an ε–small shear transformation in p we get rid of terms that are linear in p, and scaling p by $\sqrt{\varepsilon}$ we arrive at

$$H_\varepsilon(x, \hat{y}, p, q; y^*, \omega) \;=\; \langle \omega, \hat{y} \rangle \;+\; \varepsilon \mathcal{H}(p, q; y^*, \omega) \;+\; \mathcal{O}(\varepsilon^2) \tag{8}$$

with

$$\mathcal{H}(p, q; y^*, \omega) \;=\; \frac{a(y^*)}{2}p^2 \;+\; V_{y^*}(q) \;,$$

compare with (4).

Here V_{y^*} can be interpreted as an n–parameter family of 1–dimensional potentials, and critical points q^* of V_{y^*} correspond to invariant n–tori $\mathbb{T}^n \times \{(0, 0, q^*; y^*, \omega)\}$ of the 'intermediate' integrable system with Hamiltonian $\mathcal{H}_\varepsilon = \langle \omega, \hat{y} \rangle + \varepsilon \mathcal{H}$. The $y^* \in \mathbb{R}^n \cong \mathbb{Y}$ around which $\hat{y}$ is localized has the character of a (distinguished) parameter; let us make this explicit by writing

$$\mu \;:=\; y^* \;.$$

While critical points of a single potential are generically non-degenerate, it is a generic property for the n–parameter family V_μ of potentials to encounter critical points up to co-dimension n. Note that this amounts to a genericity condition on the perturbation H_1 of H_0. More precisely, the μ–values parametrising a potential V_μ with a degenerate critical point of co-dimension k form an $(n - k)$–dimensional submanifold Λ_k in μ–space.

Lemma 2.2 *[**Versality**] In the above circumstances, let $\mu^* \in \Lambda_k$ and put $d = k + 2$. Then all derivatives at q^* of order $j < d$ vanish, so*

$$V_{\mu^*}(q) \;=\; \frac{b(\mu^*)}{d!}(q - q^*)^d \;+\; \mathcal{O}\left((q - q^*)^{d+1}\right) \tag{9}$$

with $b(\mu^*) \neq 0$ and $2 \leq d \leq n + 2$ near the critical point q^* of V_{μ^*}. When $d = 2$ the critical point q^* is non-degenerate and $(p, q) = (0, q^*)$ is a non-degenerate equilibrium of the one-degree-of-freedom system — a saddle if $ab < 0$ and a centre if $ab > 0$. In case $d \geq 3$ the equilibrium is parabolic and it is furthermore generic for the family V_μ to provide a versal unfolding of the degenerate critical point. Also this genericity condition is a condition on the perturbation H_1.

Let us translate co-ordinates to $q^* = 0$ and concentrate on a degenerate critical point with $d \leq n+1$. Then

$$\mathcal{H}(p, q; \mu, \omega) \;=\; \frac{a(\mu)}{2} p^2 \;+\; \frac{b(\mu)}{d!} q^d \;+\; \sum_{j=1}^{d-2} \frac{c_j(\mu)}{j!} q^j \;+\; \mathcal{O}(q^{d+1})$$

and $\mathcal{H}_\varepsilon = \langle \omega, \hat{y} \rangle + \varepsilon \mathcal{H}$ has the form (1.4) of [5] with $\lambda_j = \varepsilon c_j(\mu)$. Thus under small perturbation, satisfying the above genericity conditions, the resonant Lagrangean torus leads to the entire bifurcation scenario detailed in [5]. This amounts to the classical hierarchy of the cuspoids [27, 3] unfolding the singularities A_{k+1}, $k \in \mathbb{N}$, that are Cantorised in the now familiar way by taking out a small neighbourhood of the dense set of resonances mentioned before and using KAM Theory. Granted the proof of Lemma 2.2 (given below), this proves Theorem 2.1. $\square$

Remark 2.3 Since $\mu = y - \hat{y}$ was introduced by localization the rôle of the unfolding parameter λ is ultimately played by the action y conjugate to the toral angles, provided that $c : \mu \mapsto c(\mu)$ is at $\mu = 0$ a submersion from $\mathbb{R}^n$ to $\mathbb{R}^{d-2}$. The tori $\mathbb{T}^n \times \{(0, 0, q^*)\}$ with $d = n+2$ are generically isolated and may therefore disappear in a resonance gap. In case $n - d + 2 \geq 1$ Diophantine approximation of dependent quantities does yield persistence on Cantor sets, see [8, 5, 20] for more details.

Proof of Lemma 2.2. For a description of the compact-open topology on holomorphic extensions of real analytic systems we refer to the introduction. We recall that Singularity Theory in the real analytic setting coincides with that in the C^k setting for k large. Therefore the families V_μ are also versal in the real analytic setting.
The versality of the family V_μ of potentials amounts to a genericity condition on the same family. This in turn is implied by a genericity condition on H_1 since the mapping $H_1 \mapsto \bar{H}_1 \mapsto V_\mu$ is a composition of submersions. Indeed, the latter part $\bar{H}_1 \mapsto V_\mu$ is merely a scaling in p after which the quadratic part $\frac{a}{2}p^2$ is split off by means of the Morse Lemma [22]. The normalization $H_1 \mapsto \bar{H}_1$ consists of a co-ordinate transformation followed by truncation of higher order terms which for sufficiently small ε has the character of a (linear) projection. $\square$

Example 2.4 [**Unfolding a degenerate minimum**] We consider the case $d = 4$ for which the one-degree-of freedom family has the form

$$\mathcal{H}(p, q; \mu, \omega) \;=\; \frac{1}{2} p^2 \;+\; \frac{1}{24} q^4 \;+\; \lambda_1(\mu) q \;+\; \frac{\lambda_2(\mu)}{2} q^2 \;, \tag{10}$$

versally unfolding the singularity A_3. Note that this example occurs persistently for $N \geq 4$ degrees of freedom. For definiteness we fix $N = 4$. So we started with four action variables $\mathbb{R}^4 = \{I_1, \ldots, I_4\}$ in which the 3-dimensional resonance hypersurface $\mathbb{Y}$ is defined by $\langle k, DH_0(I) \rangle = 0$. Locally we have the variable p transverse to $\mathbb{Y}$ and $\mu = (\mu_1, \mu_2, \mu_3)$ parametrises $\mathbb{Y}$.

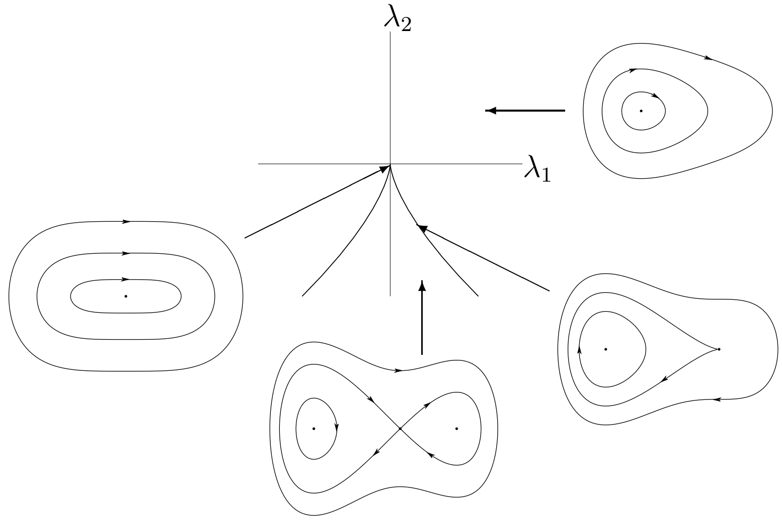

Figure 1: Organisation of the local dynamics near a degenerate minimum of the reduced Hamiltonian $\mathcal{H}$. Phase portraits show the reduced one-degree-of-freedom dynamics of (10). The interpretation for the N–degree-of-freedom system is given in the text.

In Figure 1 we show the organisation of the local dynamics in dependence of the parameters. We now describe the meaning of the phase portraits for the 4–degree-of-freedom system. All periodic orbits correspond to Lagrangean tori and equilibria to 3–dimensional tori. These are elliptic in the case of a center, hyperbolic in the case of a saddle and parabolic in the (two) remaining cases.

We recall that $\mu \mapsto \lambda(\mu)$ is a (local) submersion. Therefore Cantorisation in the (μ_1, μ_2, μ_3)–direction amounts to the following.

1. We begin with the cases corresponding to parabolic tori.
   (a) The central point $\lambda = 0$ corresponds to a line that Cantorises to a (real) Cantor set (*i.e.* of topological dimension 0) of Hausdorff dimension 1.
   (b) The fold lines emanating from $\lambda = 0$ correspond to planes that Cantorise to a (real) Cantor set as well, now of Hausdorff dimension 2.
2. The hyperbolic tori are parametrised over the open region in between the fold lines, which corresponds to an open 3–dimensional set. Cantorisation leads to a union of lines, smoothly parametrised over the 2–dimensional Cantor set mentioned above and also ending there. Recall that colloquially we call this a Cantor family of closed half lines; the Hausdorff dimension is 3.
3. Both open regions in Figure 1 parametrise elliptic tori, observing that in between the fold lines each point corresponds to two elliptic tori, and in the other region only to one.
   (a) In the latter case Cantorisation leads to a (real) Cantor set.

(b) In the former case Cantorisation leads to two layers of (real) Cantor sets.

In both cases the Cantor set has Hausdorff dimension 3.

As usual, Cantorisation of the librational Lagrangean tori takes place along a Cantor family of lines, of Hausdorff dimension 4. Note that this also uses the p variable to obtain the 4–dimensional (μ, p)–set. Here, and in the earlier cases, the corresponding Hausdorff measure is positive and, in fact, even close to full measure when the perturbation is small.

As in the example of the quasi-periodic centre-saddle bifurcation, we can give asympotic estimates based on the geometry sketched in Figure 1. For instance, along the symmetry axis $\lambda_1 = 0$ the distance between the two elliptic tori is of order $\sqrt{-\lambda_2(\mu)}$ as $\mu \to 0$. $\triangle$

It is instructive to compare the above results with the corresponding statements in [17]. When d is odd the bifurcation diagram contains empty regions and regions with both elliptic and hyperbolic tori. When d is even all open regions of the bifurcation diagram Cantor-parametrise at least one elliptic torus if $b > 0$, see Figure 1, while $b < 0$ yields deformations with hyperbolic tori. For a similar approach in the latter case (not using Lindstedt series) see [30].

For small perturbations εH_1 of H_0 Theorem 2.1 allows to recover what [13, 14] state in this situation. Indeed, the angular variable q takes values in $\mathbb{T}$ and on this compact set each potential V_μ, μ fixed, assumes minimum and maximum. It is a genericity condition on H_1 for these to be different from each other, *i.e.* $V_\mu \neq$ *const* for all μ, and furthermore for V_μ to assume the form (9) with d even and $b(\mu) < 0$ at a maximum q^* while $b(\mu) > 0$ at a minimum; let us denote the latter by q_*. For definiteness we concentrate on $a(\mu) > 0$ (which may always be achieved by reversing time if necessary). Then the maximum q^* corresponds to a 'hyperbolic type' torus $\mathbb{T}^n \times \{(0, 0, q^*)\}$, truly hyperbolic if the maximum is non-degenerate and parabolic otherwise. In both cases we have persistence for ω satisfying the Diophantine conditions (2) with N replaced by n. This leads to Cantorisation.

The persistence of at least one n–torus from the H_0–resonant $\mathbb{T}^n \times \{(0, 0)\} \times \mathbb{T}$ for Diophantine ω had already been established in [13], and without any genericity condition on the perturbation. What our approach adds to this is a precise description how occurring degenerate maxima of the potential, called 'of weaker persistency' in [13], lead to a Cantorised bifurcation scenario of the corresponding n–tori. This latter result cannot be obtained without genericity conditions. The generality of the result in [13] does not exclude perturbations that are rather pathological. Correspondingly, that approach does not allow to obtain information on the fine structure where the n–tori fail to be truly hyperbolic.

The minima q_* of V_μ are treated in [14]. In the non-degenerate case these correspond to elliptic n–tori, whence normal-internal resonances have to be avoided as well. Therefore, persistence of a second n–torus is obtained in [14] only on a smaller (though still measure-theoretically large) subset S. For generic perturbations we can now explain the fine structure. In particular the tori coming from degenerate minima q_* do not have to be excluded. In fact, these give the opportunity to enlarge S a bit. For instance, when $d = 4$ in (9) we recover the bifurcation diagram given in Figure 1 and next to at least one centre for all nonzero $\lambda \in \mathbb{R}^2$ we have an additional saddle in between the two fold lines; hence, here only Diophanticity of internal frequencies is needed to obtain persistence of a second family of invariant n–tori in the resonant zone.

Remark 2.5 Without the genericity conditions on the perturbation there may be tori $\mathbb{T}^n \times \{(0, 0, q^*)\}$ with $d > n + 2$. In an attempt to still apply the results of [5] we may introduce extra (external) parameters ν to provide a versal unfolding. The perturbed system then displays again the Cantorised bifurcation scenario and contains the original perturbed system as a subsystem. The torus $\mathbb{T}^n \times \{(0, 0, q^*)\}$ is an invariant torus of the intermediate system. However, we cannot expect this torus to be present in the perturbed system since this torus is moved by the perturbation. We still have the weaker conclusion, though, that the Cantorised family of n–tori contains parabolic tori of at most degeneracy d (as found in the intermediate system).

Note that this approach does not allow us to drop the genericity conditions that we had to impose on the perturbation when recovering the results of [13, 14]. Indeed, the small constant ε in Theorem 2.1 of [5] depends on d and may tend to 0 as $d \to \infty$. Since the results in [13, 14] are valid for non-generic perturbations as well one may speculate that the latter does not occur.

Example 2.6 [**A non-versal perturbation**] The class of perturbed Hamiltonians

$$H_\varepsilon(\varphi, I) \;=\; H_0(I) \;+\; \varepsilon H_1(\varphi) \tag{11}$$

figures a perturbation that is independent of the action variable I. A single resonance (7) leads again to a reduced system (4) of the form

$$\mathcal{H}(p, q; \mu) \;=\; \frac{a(\mu)}{2} p^2 \;+\; V(q)$$

where the potential V is now equal to the $\mathbb{T}^n$–average $\bar{H}_1$ of H_1, with no further transformations. As a μ–dependent family this potential is trivial, being the same for all parameter values. Invariant n–tori correspond to $(p, q) = (0, q^*)$ with $V'(q^*) = 0$ and if $b := V''(q^*) \neq 0$ the triviality of the family is not problematic. Indeed, the torus is elliptic for $ab > 0$ and hyperbolic for $ab < 0$ with no need for an unfolding. On the other hand, if $b = 0$ then (11) is a very degenerate system: the torus is parabolic and the potential cannot provide the necessary unfolding. Remark 2.5 still applies, though. $\triangle$

The degeneracy in Example 2.6 that occurs for $b = 0$ should then be seen as a warning sign that the model (11) is problematic and might need to be changed. This kind of warning sign is given whenever Theorem 2.1 does not apply, cf. [27]. The necessary genericity conditions can be explicitly checked in examples and provide one with clues of what exactly is happening. For instance, where the n–parameter family V_μ of potentials encounters critical points of co-dimension exceeding n the perturbed system (4) deserves further examination.

This might result in an adjusted model for what one is trying to describe. Another possible outcome is that a symmetry is found in (4), and that within the 'symmetric universe' the co-dimension no longer exceeds n. The unfolding provided by V_μ then is expected to be versal within the 'symmetric universe', see [24], and also the $\mathcal{O}(\varepsilon^2)$–terms in (8) do not break the symmetry. Also other reasons for seemingly non-generic behaviour are known to exist, see *e.g.* [21] for the persistent occurrence of a degenerate bifurcation.

A different approach (that avoids to impose genericity conditions) is persued in [16], where periodic orbits foliating invariant 2–tori are searched for using the zeroes of the subharmonic Mel'nikov function. In case the latter vanishes identically, a second order Mel'nikov function is defined with similar properties, and so on. If all higher order Melnikov functions vanish identically, then the whole torus consisting of periodic orbits is shown to survive the perturbation.

3 An umbilic example

To understand how results similar to those of the previous section can be obtained if the Kolmogorov condition (6) is replaced by Rüssmann's non-degeneracy condition, we now consider the following example in $N = 5$ degrees of freedom. Starting point remains the perturbed Hamiltonian (3) with $I = (y, p)$, and for the unperturbed part we work with

$$H_0(y,p) \;=\; \sum_{i=1}^{4} \mathrm{e}^{i-1} y_i + \frac{1}{2} y_i^2 + \frac{1}{6} y_i^3 \;+\; \frac{1}{6} p^3 \;, \tag{12}$$

where we use that the vector $(1, \mathrm{e}, \mathrm{e}^2, \mathrm{e}^3)$ is Diophantine. Then Rüssmann's non-degeneracy condition

$$\mathbb{R}^5 \;=\; < \frac{\partial^{|\ell|}\omega}{\partial I^\ell} \;\Big|\; 0 \neq \ell \in \mathbb{N}_0^5 > \;\supseteq\; < \frac{\partial\omega}{\partial y_1}, \frac{\partial\omega}{\partial y_2}, \frac{\partial\omega}{\partial y_3}, \frac{\partial\omega}{\partial y_4}, \frac{\partial^2\omega}{\partial p^2} >$$

that the partial derivatives span the frequency space is satisfied everywhere. We are interested in the fate of the resonant tori $p = 0$. Again we apply a normalizing transformation that turns the perturbed Hamiltonian $H_\varepsilon = H_0 + \varepsilon H_1$ into

$$H_\varepsilon(x, \hat{y}, p, q; \mu, \omega) \;=\; H_0(\hat{y}, p; \mu, \omega) \;+\; \varepsilon \bar{H}_1(\hat{y}, p, q; \mu, \omega) \;+\; \mathcal{O}(\varepsilon^2)$$

where $\bar{H}_1$ is the $\mathbb{T}^4$–average along x of H_1 at $\varepsilon = 0$.

A vanishing 1–jet (in the (p, q)–variables) of a Hamiltonian function merely amounts to $(p, q) = (0, 0)$ being a relative equilibrium. This is already true for H_0, and to achieve this for H_ε we (again) translate co-ordinates to $q^* = 0$. Using the p^3–term in H_0 a translation in p allows to remove the p^2–term in the expansion of $\bar{H}_1$ in p and q. In case the coefficient $a_{02}(0; \mu, \omega)$ of q^2 does not vanish at $\mu = 0$ we scale p by $\sqrt{\varepsilon}$ and q by $\sqrt[4]{\varepsilon}$ to recover the quasi-periodic centre-saddle bifurcation encountered in the previous section. Finally, for nonzero $a_{11}(0; 0, \omega) pq$ we scale p and q both by ε, revealing the relative equilibrium to be hyperbolic. In the expansion

$$\bar{H}_1 \;=\; \sum_{k+l=3} \frac{a_{kl}(\hat{y}; \mu, \omega)}{k!\, l!} p^k q^l \;+\; h.o.t.$$

we therefore start with third order terms. We emphasise that for generic perturbations H_1 this cannot be avoided to occur at 1–parameter subfamilies.

Shrinking $\mathbb{Y}$ a bit, if necessary, the coefficient[2] a_{03} is bounded away from zero. Scaling p by $\varepsilon^{\frac{2}{3}}$ and q by $\varepsilon^{\frac{1}{3}}$ we obtain

$$H_\varepsilon(x, \hat{y}, p, q; \mu, \omega) \;=\; \langle \omega, \hat{y} \rangle \;+\; \varepsilon^2 \mathcal{H}(p, q; \mu, \omega) \;+\; \mathcal{O}(\varepsilon^{\frac{7}{3}})$$

with

$$\mathcal{H}(p, q; \mu, \omega) \;=\; \frac{a}{6} p^3 \;+\; \frac{b}{6} q^3 \;+\; c_1(\mu) q \;+\; c_2(\mu) p \;+\; c_3(\mu) pq$$

where $a \approx 1$, $b = 6a_{03}(0; \mu, \omega)$. The coefficient functions c_1, c_2 and c_3 vanish at $\mu = 0$, this allows to rescale a_{01} by $\varepsilon^{\frac{2}{3}}$ to yield $c_1(\mu)$ and a_{10} by $\varepsilon^{\frac{1}{3}}$ to yield $c_2(\mu)$; a_{11} is not rescaled, we simply put $c_3 = a_{11}(0; \mu, \omega)$. Note that we do not obtain the unfolding of D_4 used in [6, 20], but a form adapted to the critical singularity $\frac{a}{6}p^3 + \frac{b}{6}q^3$, and that this form leads to the hyperbolic umbilic. Still, we conclude that the family of resonant Lagrangean tori $p = 0$ of (12) may lead to umbilical torus bifurcations of invariant 4–tori.

4 Rüssmann Hamiltonians

The example of the previous section looks in one aspect quite degenerate — the resonance $p = 0$ coincides with the hypersurface $p = 0$ where the Kolmogorov condition (6) fails. In general the (transverse) intersection of these should be a co-dimension 2 submanifold, and singularities of the equation $\det D^2 H_0 = 0$ may lead to further complications. For instance, one could consider the example in the previous section with one more degree of freedom and add[3] the terms

$$\mathrm{e}^4 y_5 \;+\; \frac{1}{2} y_5^2 \;+\; \frac{1}{6} y_5^3 \;+\; \frac{1}{2} y_5 p^2$$

to the unperturbed Hamiltonian (12). Then the analysis of the previous section concerns $y_5 = 0$ and the obvious question is how the perturbed system behaves when unfolded by y_5 — scaled by $\varepsilon^{\frac{2}{3}}$. Note that it is generic for H_0 to satisfy some form of Rüssmann non-degeneracy at every point, cf. [29].

Leaving such complications aside for the moment, a 'naive' generalization of the example in the previous section leads to

$$H_0(\hat{y}, p; \mu, \omega) \;=\; \langle \omega, \hat{y} \rangle \;+\; \frac{a(\mu)}{\ell!} p^\ell$$

with $a(\mu) \neq 0$ and $2 \leq \ell \leq n + 2$. For instance, if $\ell = 3$ the potential (9) leads to the quasi-periodic centre-saddle bifurcation (unfolding the singularity A_2) when $d = 2$. For

[2]Here we depart from the general theory of planar singularities that allows to transparently treat the relative equilibria. Indeed, the co-ordinates p and q already have a 'meaning', so we had to sharpen the usual assumption that the (homogeneous) 3–jet does not have multiple roots.

[3]Here we use that the vector $(1, \mathrm{e}, \ldots, \mathrm{e}^4)$ is Diophantine as well.

$d = 3$ this similarly leads to umbilic tori (unfolding the singularity D_4) and to the simple singularities E_6 and E_8 (see [19, 20]) when $d = 4$ and 5, respectively.

The umbilic example of the previous section gives some confidence that it should still be possible to find adapted scalings that turn $\frac{a}{\ell!}p^\ell + \varepsilon \bar{H}_1$ for generic perturbation H_1 into versal unfoldings of occuring singularities. Note, however, that the special 'starting point' $\frac{a}{\ell!}p^\ell$ leads to a classification that may slightly differ from the classification of one-degree-of-freedom equilibria by means of planar singularities.

5 Conclusions

Summarizing we may state that the for a *single* resonance the Kolmogorov condition (6) restricts the occurring singularities to the family A_{k+1} of corank–1–singularities (9), while under the weaker Rüssmann condition also singularities of corank 2 may occur. Theorem 2.1 treats the former situation, while the example in Section 3 concerns the latter case. All these singularities describe the normal behaviour of Cantor bundles of (degenerate) tori, with Cantorised unfoldings.

The corank–2–singularities D_4, D_5, D_6 and E_6 of low co-dimension are still simple and E_8 and the complete family D_{k+1}, $k \geq 3$, unfolded by the umbilics, are simple as well. Next to these also singularities with modal parameters become possible, leading for high N to all quasi-periodic bifurcations of [20]. We note that, for these bifurcations to take their standard form in a resonance gap, a scaling will be needed.

In the case of an m–fold resonance the above normalization procedure applied to (3) leads to an $(N-m)$–parameter family of Hamiltonian systems defined on $\mathbb{T}^m \times \mathbb{R}^m$. Here, non-degenerate minima correspond to elliptic $(N-m)$–dimensional tori [15, 12, 28, 10]. For these cases there exist many results on quasi-periodic persistence, employing KAM Theory.

In the spirit of the present paper, one should consider degenerate minima as well. Under the Kolmogorov condition (6) this leads to corank–m–singularities. The corresponding quasi-periodic bifurcation theory still has to be developed. Under the Rüssmann condition similarly this gives rise to singularities of corank $2m$. We like to stress that none of these complications can be avoided.

Acknowledgements. We thank an anonymous referee for challenging us with the example $H(\varphi, I) = H_0(I) + \epsilon H_1(\varphi)$ of a non-versal perturbation.

References

[1] *V.I. Arnol'd : Geometrical Methods in the Theory of Ordinary Differential Equations* ; Springer (1983)

[2] *V.I. Arnol'd, V.V. Kozlov and A.I. Neishtadt ;* Mathematical Aspects of Classical and Celestial Mechanics ; in *Dynamical Systems III* (ed. V.I. Arnol'd) Springer (1988)

[3] *V.I. Arnol'd, V.A. Vasil'ev, V.V.Goryunov and O.V. Lyashko ;* Singularity theory I : Singularities, local and global theory ; in *Dynamical Systems VI* (ed. V.I. Arnol'd) Springer (1993)

[4] *H.W. Broer, M.C. Ciocci, H. Hanßmann and A. Vanderbauwhede :* Quasi-periodic stability of normally resonant tori ; *Physica D* **238**(3), p. 309 – 318 (2009)

[5] *H.W. Broer, H. Hanßmann and J. You :* Bifurcations of normally parabolic tori in Hamiltonian systems ; *Nonlinearity* **18**, p. 1735 – 1769 (2005)

[6] *H.W. Broer, H. Hanßmann and J. You :* Umbilical Torus Bifurcations in Hamiltonian Systems ; *J. Diff. Eq.* **222**, p. 233 – 262 (2006)

[7] *H.W. Broer, J. Hoo and V. Naudot :* Normal linear stability of quasi-periodic tori ; *J. Diff. Eq.* **232**(2), p. 355 – 418 (2007)

[8] *H.W. Broer, G.B. Huitema and M.B. Sevryuk :* *Quasi-Periodic Motions in Families of Dynamical Systems : Order amidst Chaos* ; LNM **1645**, Springer (1996)

[9] *H.W. Broer, G.B. Huitema and F. Takens :* Unfoldings of quasi-periodic tori ; *Mem. AMS* **83** #421, p. 1 – 82 (1990)

[10] *H.W. Broer and M.B. Sevryuk :* KAM Theory : quasi-periodicity in dynamical systems ; p. 249 – 344 in : *Handbook of Dynamical Systems,* Volume 3 (eds. H.W. Broer, B. Hasselblatt and F. Takens) North-Holland (2010)

[11] *H.W. Broer and F.M. Tangerman :* From a differentiable to a real analytic perturbation theory, applications to the Kupka Smale theorems ; *Ergod. Th. & Dynam. Sys.* **6**, p. 345 – 362 (1986)

[12] *F. Cong, T. Küpper, Y. Li and J. You :* KAM-Type Theorem on Resonant Surfaces for Nearly Integrable Hamiltonian Systems ; *J. Nonlinear Sci.* **10**, p. 49 – 68 (2000)

[13] *C.-Q. Cheng :* Birkhoff–Kolmogorov–Arnold–Moser tori in convex Hamiltonian systems ; *Comm. Math. Phys.* **177**, p. 529 – 559 (1996)

[14] *C.-Q. Cheng :* Lower Dimensional Invariant Tori in the Regions of Instability for Nearly Integrable Hamiltonian Systems ; *Comm. Math. Phys.* **203**, p. 385 – 419 (1999)

[15] *C.-Q. Cheng and S. Wang :* The Surviving of Lower Dimensional Tori from a Resonant Torus of Hamiltonian Systems ; *J. Diff. Eq.* **155**(2), p. 311 – 326 (1999)

[16] *L. Corsi and G. Gentile :* Melnikov theory to all orders and Puiseux series for subharmonic solutions ; *J. Math. Phys.* **49**(112701), p. 1 – 29 (2008)

[17] *G. Gallavoti, G. Gentile and A. Giuliani :* Fractional Lindstedt series ; *J. Math. Phys.* **47**(012702), p. 1 – 33 (2006)

[18] *H. Hanßmann :* The Quasi-Periodic Centre-Saddle Bifurcation ; *J. Diff. Eq.* **142**(2), p. 305 – 370 (1998)

[19] *H. Hanßmann :* Hamiltonian Torus Bifurcations Related to Simple Singularities ; p. 679 – 685 in *Dynamic Systems and Applications, Atlanta 2003* (eds. G.S. Ladde, N.G. Medhin and M. Sambandham) Dynamic Publishers (2004)

[20] *H. Hanßmann : Local and Semi–Local Bifurcations in Hamiltonian Dynamical Systems — Results and Examples* ; LNM **1893**, Springer (2007)

[21] *H. Hanßmann and B. Sommer :* A degenerate bifurcation in the Hénon–Heiles family ; *Cel. Mech. & Dyn. Astr.* **81**(3), p. 249 – 261 (2001)

[22] *M.W. Hirsch : Differential topology* ; Springer (1976)

[23] *L. Markus and K.R. Meyer :* Generic Hamiltonian dynamical systems are neither integrable nor ergodic ; *Mem. AMS* #144, p. 1 – 52 (1974)

[24] *V. Poénaru : Singularités* C^∞ *en Présence de Symétrie* ; LNM **510**, Springer (1976)

[25] *J. Pöschel :* Integrability of Hamiltonian Systems on Cantor Sets ; *Comm. Pure Appl. Math.* **35**, p. 653 – 696 (1982)

[26] *H. Rüssmann :* Invariant tori in non-degenerate nearly integrable Hamiltonian systems ; *Regular Chaotic Dynamics* **6**, p 119 – 204 (2001)

[27] *R. Thom : Structural Stability and Morphogenesis. An Outline of a General Theory of Models*, 2nd ed,; Addison-Wesley (1989)

[28] *M.B. Sevryuk :* The classical KAM theory at the dawn of the twenty-first century ; *Moscow Math. J.* **3**(3), p. 1113 – 1144 (2003)

[29] *J. Xu and J. You :* A Note on the paper “Invariant tori for nearly integrable Hamiltonian systems with degeneracy”; preprint, Nanjing University (2006)

[30] *J. You :* A KAM Theorem for Hyperbolic-Type Degenerate Lower Dimensional Tori in Hamiltonian Systems ; *Comm. Math. Phys.* **192**(1), p. 145 – 168 (1998)