Quasi-periodic solutions and stability of the equilibrium for quasi-periodically forced planar reversible and Hamiltonian systems under the Bruno condition

Heinz Hanßmann and Jianguo Si*

Mathematisch Instituut, Universiteit Utrecht
Postbus 80.010, 3508 TA Utrecht, The Netherlands

School of Mathematics, Shandong University
Jinan, Shandong 250100, People's Republic of China

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Abstract. In this paper we investigate the existence of quasi-periodic solutions of non-autonomous two-dimensional reversible and Hamiltonian systems under the Bruno condition. As an application we study the dynamical stability of the trivial solution at the origin of a quasi-periodically forced planar system. Under a mild non-degeneracy condition we give a criterion that is necessary and sufficient for a large class of systems.

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1 Introduction

We are concerned with quasi-periodic solutions and stability of $\{I = 0\}$ for the non-autonomous system of two differential equations

$$\dot{I} = R(I, \varphi, t, a), \quad \dot{\varphi} = a + Q(I, \varphi, t, a), \tag{1.1}$$

where a is a one-dimensional parameter and $R(0, \varphi, t, a) = Q(0, \varphi, t, a) = 0$. Furthermore R and Q are real analytic in all variables and quasi-periodic in t and φ with basic frequencies $\Omega_1 = (\omega_1, \omega_2 \ldots, \omega_{N_1})$ and $\Omega_2 = (\omega_{N_1+1}, \omega_{N_1+2} \ldots, \omega_{N_1+N_2})$, respectively.

In the analysis of dynamical systems, the search for equilibria and their stability is one of the first tasks to be carried out, since they organize the phase flow to a large extent. Determining the stability of the equilibrium, in general, is not an easy task as it usually requires finding a Lyapunov function. The theory for linear stability is well established and easy to apply, whence the first step in analysing stability within a nonlinear system is to

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E-mail address: sijgmath@sdu.edu.cn or sijgmath@yahoo.com.cn

linearize it. If at least one eigenvalue has positive real part, the equilibrium is unstable for both the linearized system and the (nonlinear) full system. Unfortunately, the converse is not true; the higher order terms may destroy the stability of the linear part, and a different method must be used. When the vectorfield is periodic in t, the stability depends on the eigenvalues of the Poincaré mapping of the linearized system and the stability of the periodic orbit of the full system can be determined by various perturbation methods. For example, in the hyperbolic case, the periodic orbit of the full system is not stable. In the elliptic case, if the eigenvalues of the Poincaré mapping of the linearized system satisfy some non-resonant relations, one can prove via Moser's twist theorem that the periodic orbit of the full system is stable. For a detailed description see [11, 18, 13, 7, 8, 12] and references therein. The methods used to study the stability of the trivial solution $(x(t), y(t)) \equiv (0, 0)$ when the system is periodic in t cannot be applied directly to the quasi-periodic case. When system (1.1) is a Hamiltonian system (or a reversible one) which depends on time in a quasi-periodic way and its frequency vector satisfies a Diophantine condition, Yu.N. Bibikov in [2] proves the existence of quasi-periodic solutions of (1.1) and applies it to study the stability of $\{(x, y) = (0, 0)\}$ for the Hamiltonian system

$$\dot{x} = \frac{\partial E}{\partial y}, \quad \dot{y} = -\frac{\partial E}{\partial x} \tag{1.2}$$

with frequency vector $\Omega_1 = (\omega_1, \dots, \omega_{N_1})$ and Hamiltonian function

$$E = \frac{y^2}{2} + \frac{x^{2n+2}}{2n+2} + F(x, y, \Omega_1 t)$$

and the reversible system

$$\ddot{x} + x^{2n+1} = f(x, \dot{x}, \Omega_1 t)$$

(under some reasonable assumptions on F and f), respectively. Recently, Liu [9] studied the stability of the equilibrium of the reversible system

$$\dot{x} = a_0(t)y^{2m+1} + f_1(x, y, t), \quad \dot{y} = -b_0(t)x^{2n+1} + f_2(x, y, t) \tag{1.3}$$

and a similar Hamiltonian system using Bibikov's results. In the present paper, we also assume that the system (1.1) is a reversible system or a Hamiltonian system. Motivated by the results of Bibikov [2] and Rüssmann [16, 17], substituting the Bruno condition for the Diophantine condition we construct in Section 2 a family of solutions of (1.1) which are quasi-periodic in both the time variable and the angle variable with basic frequencies Ω_1 and Ω_2. In Section 3 we apply this to study the stability of the equilibrium of the planar system (3.1), which has both (1.2) and (1.3) as special cases. We first introduce some definitions and notations.

Definition 1.1 *(Bruno condition) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $1 = q_0 < q_1 \cdots$ be the main denominators of the continued fraction of α, we say that α satisfies the Bruno condition if*

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty.$$

Definition 1.2 [16] *A continuous function* $\Phi : [0,\infty] \to \mathbb{R}$ *is called an approximation function if*

1. $1 = \Phi(0) \geq \Phi(s) \geq \Phi(t) > 0$ *for* $0 \leq s < t < \infty$,
2. $\Phi(1) = 1$, *implying that* $\Phi(s) = 1$ *for* $0 \leq s \leq 1$,
3. $s^{\lambda}\Phi(s) \to 0$ *as* $s \to \infty$ *for any* $\lambda \geq 0$, *implying that* $\Phi^{\lambda\mu} := \sup_{s\geq 1} s^{\lambda}\Phi(s)^{\frac{1}{\mu}} < \infty$ *for all* $\lambda \geq 0, \mu > 0$,
4. $\displaystyle\int_1^{\infty} \log \frac{1}{\Phi(s)} \frac{ds}{s^2} < \infty$.

Remark 1.3 As stated in [16], the consequent use of a general approximation function as defined in Definition 1.2 makes many of the necessary calculations simple and quite natural. While condition 2 in Definition 1.2 is unessential but technically useful, the condition 3 is necessary in order to ensure that the set (2.25) is of small measure whereas the condition 4 is equivalent to the Bruno condition in the case of the plane. This equivalence and the existence of an approximation function satisfying Definition 1.2 can be found in [16, 17]. The Bruno condition is necessary in many problems, see [19, 14, 20, 3]. Moreover, as an example in [5] shows the Bruno condition is strictly weaker than the Diophantine condition.

Throughout this paper, we write

$$\Omega = (\Omega_1, \Omega_2) = (\omega_1, \omega_2, \ldots, \omega_{N_1}, \omega_{N_1+1}, \omega_{N_1+2}, \ldots, \omega_{N_1+N_2})$$

and assume that there is a $\gamma > 0$ for which Ω satisfies the Bruno condition

$$|\langle k, \Omega\rangle| \geq \gamma\Phi(|k|) \quad \text{for all } 0 \neq k \in \mathbb{Z}^{N_1+N_2} \tag{1.4}$$

where Φ is an approximation function as defined in Definition 1.2. We use the inner product

$$\langle k, \Omega\rangle = \langle k_1, \Omega_1\rangle + \langle k_2, \Omega_2\rangle = \sum_{v=1}^{N_1} \tilde{k}_v\omega_v + \sum_{\mu=1}^{N_2} \tilde{k}_{N_1+\mu}\omega_{N_1+\mu}$$

where

$$k = (k_1, k_2), \quad k_1 = (\tilde{k}_1, \ldots, \tilde{k}_{N_1}), \quad k_2 = (\tilde{k}_{N_1+1,}, \ldots, \tilde{k}_{N_1+N_2})$$

and $|k| = \sum_{v=1}^{N_1} |\tilde{k}_v| + \sum_{\mu=1}^{N_2} |\tilde{k}_{N_1+\mu}|$.

Definition 1.4 *A real analytic function of two variables* $f = f(t,\varphi)$ *is called quasi-periodic with frequencies* $\Omega_1 = (\omega_1, \omega_2, \ldots, \omega_{N_1})$ *and* $\Omega_2 = (\omega_{N_1+1}, \omega_{N_1+2}, \ldots, \omega_{N_1+N_2})$ *if it can be represented as a Fourier series of the type*

$$f(t,\varphi) = \sum_{(k_1,k_2)\in\mathbb{Z}^{N_1}\times\mathbb{Z}^{N_2}} f_{k_1k_2} e^{i\langle k_1,\Omega_1\rangle t + i\langle k_2,\Omega_2\rangle\varphi}. \tag{1.5}$$

The set of all quasi-periodic real analytic functions with frequencies Ω_1 *and* Ω_2 *is denoted by* $Q(\Omega_1, \Omega_2)$.

The Fourier coefficients $f_{k_1k_2}$ in (1.5) yield the shell function

$$F(\theta,\widetilde{\theta})=\sum_{(k_1,k_2)\in\mathbb{Z}^{N_1}\times\mathbb{Z}^{N_2}} f_{k_1k_2}e^{i\langle k_1,\theta\rangle+i\langle k_2,\widetilde{\theta}\rangle} \tag{1.6}$$

for f which is 2π-periodic in each component of the two vectors $\theta=(\theta_1,\ldots,,\theta_{N_1})$ and $\widetilde{\theta}=(\widetilde{\theta}_{N_1+1},\ldots,\widetilde{\theta}_{N_1+N_2})$, and satisfies $f(t,\varphi)\equiv F(\Omega_1 t,\Omega_2\varphi)$ identically in t and φ. From the shell function $F:\mathbb{T}^{N_1}\times\mathbb{T}^{N_2}\longrightarrow\mathbb{R}$ (where $\mathbb{T}=\mathbb{R}/(2\pi\mathbb{Z})$) the inverse Fourier transformation

$$f_{k_1,k_2}=\frac{1}{(2\pi)^{N_1+N_2}}\int_{\mathbb{T}^{N_1+N_2}}F(\theta,\widetilde{\theta})e^{-i(\langle k_1,\theta\rangle+\langle k_2,\widetilde{\theta}\rangle)}d\theta d\widetilde{\theta}$$

yields back the (Fourier coefficients of the) quasi-periodic function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$. We denote by

$$F^K(\theta,\widetilde{\theta})=\sum_{|k|\le K} f_{k_1,k_2}e^{i\langle k_1,\theta\rangle+i\langle k_2,\widetilde{\theta}\rangle} \tag{1.7}$$

the truncation of the Fourier series (1.6) of F up to order K. Because of real analyticity the shell function is bounded in a complex neighbourhood of $\mathbb{T}^{N_1+N_2}$ defined by inequalities $|\operatorname{Im}\theta_i|\le r$, $|\operatorname{Im}\widetilde{\theta}_j|\le r$ for some $r>0$. For fixed $r>0$ we denote by $Q_r(\Omega_1,\Omega_2)\subset Q(\Omega_1,\Omega_2)$ the set of real analytic functions f such that the corresponding shell functions F are bounded on the subset $\Pi_r=\{(\theta,\widetilde{\theta})\in\mathbb{C}^{N_1}\times\mathbb{C}^{N_2}:|\operatorname{Im}\theta_i|\le r,|\operatorname{Im}\widetilde{\theta}_j|\le r\}$. Using the supremum norm

$$|f|_r=\sup_{(\theta,\widetilde{\theta})\in\Pi_r}\left|\sum_{(k_1,k_2)\in\mathbb{Z}^{N_1}\times\mathbb{Z}^{N_2}} f_{k_1k_2}e^{i\langle k_1,\theta\rangle+i\langle k_2,\widetilde{\theta}\rangle}\right|=\sup_{(\theta,\widetilde{\theta})\in\Pi_r}\left|F(\theta,\widetilde{\theta})\right|$$

on Π_r and Cauchy's formula, it follows that

$$|f_{k_1,k_2}|\le|f|_r e^{-|k|r}\quad\text{and}\quad|f_\varphi|_{r'}\le(r-r')^{-1}|f|_r. \tag{1.8}$$

for $f\in Q_r(\Omega_1,\Omega_2)$ and $0<r'<r$. For $r,s>0$ we define

$$D_{r,s}=\{I\in\mathbb{C}:|I|<s\}\times\Pi_r=\{(I,\theta,\widetilde{\theta}):|I|<s,|\operatorname{Im}\theta_i|<r,|\operatorname{Im}\widetilde{\theta}_j|<r\}$$

and for $\gamma>0$ we define the set

$$A_\gamma=\left\{a\in\mathbb{R}:|\langle k_1,\Omega_1\rangle+\langle k_2,a\Omega_2\rangle|\ge\gamma\Phi(|k|)\text{ for all }0\ne k\in\mathbb{Z}^{N_2+N_2}\right\}$$

with its complex neighbourhood $O_h:=A_\gamma+h$ of radius h and put $|\cdot|_{r,s,h}=\sup_{D_{r,s}\times O_h}|\cdot|$.

Definition 1.5 *Let $f\in Q(\Omega_1,\Omega_2)$ have the shell function F. The Fourier coefficient*

$$f_{00}=\frac{1}{(2\pi)^{N_1+N_2}}\int_{\mathbb{T}^{N_1+N_2}}F(\theta,\widetilde{\theta})d\theta d\widetilde{\theta}$$

is called the average of f and denoted by $\operatorname{av}(f)$.

For a quasi-periodic function v with rationally independent frequencies $\omega_1, \ldots, \omega_d$ and shell function V, satisfying $v(t) = V(\omega_1 t, \ldots, \omega_d t)$, the time average

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T v(t)dt$$

coincides with the space average

$$v_0 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} V(\theta)d\theta$$

and we denote the common value by $\mathrm{av}(v)$ as well.

2 Quasi-periodic solutions

In this section, we first study the existence of quasi-periodic solutions for a 2-dimensional reversible system. Then we derive a similar result in the Hamiltonian case. Our proof is based on general ideas of KAM-theory as laid out in [10, 11, 4, 15, 16, 2].

2.1 Reversible systems

Consider the reversible system

$$\begin{cases} \dot{I} = R(I, \varphi, t, a) \\ \dot{\varphi} = a + Q(I, \varphi, t, a), \end{cases} \tag{2.1}$$

with a parameter $a \in \mathbb{R}$. We assume that for fixed I and a we have $R(I, \cdot, \cdot, a), Q(I, \cdot, \cdot, a) \in Q_r(\Omega_1, \Omega_2)$ and the corresponding shell functions $\widetilde{R}(I, \widetilde{\theta}, \theta, a)$ and $\widetilde{Q}(I, \widetilde{\theta}, \theta, a)$ are real analytic on $D_{r,s} \times O_h$, satisfying

$$|\widetilde{R}|_{r,s,h} < \gamma s^2, \quad |\widetilde{Q}|_{r,s,h} < \gamma s \tag{2.2}$$

and

$$R(I, -\varphi, -t, a) = -R(I, \varphi, t, a), \quad Q(I, -\varphi, -t, a) = Q(I, \varphi, t, a) \tag{2.3}$$

for all $(I, \varphi, t, a) \in \{|I| < s\} \times \mathbb{R}^2 \times O_h$. Subsequently, we say that R is an odd function and Q is an even function if (2.3) is identically satisfied, whence (2.1) is a reversible system.

Theorem 2.1 *Under the above assumptions and if the Bruno condition (1.4) holds, there is an s^* such that for any positive $s < s^*$ there is a function $a_0 : A_\gamma \longrightarrow \mathbb{R}$ satisfying $|a_0(\alpha) - \alpha| < h$ for all $\alpha \in A_\gamma$ and a change of variables*

$$I = \varrho + v(\varrho, \psi, \theta, \alpha), \quad \varphi = \psi + u(\psi, \theta, \alpha), \quad \alpha \in A_\gamma \tag{2.4}$$

which transforms system (2.1) with $a = a_0(\alpha)$ into a system

$$\dot{\varrho} = V(\varrho, \psi, t, \alpha), \quad \dot{\psi} = \alpha + U(\varrho, \psi, t, \alpha) \tag{2.5}$$

such that

$$V(0, \psi, t, \alpha) = \frac{\partial V}{\partial \varrho}(0, \psi, t, \alpha) = U(0, \psi, t, \alpha) = 0. \tag{2.6}$$

It follows that for each ψ_0 the system (2.1) admits the quasi-periodic solution

$$I(t) = v(0, \psi_0 + \alpha t, \Omega_1 t, \alpha), \quad \varphi(t) = \psi_0 + \alpha t + u(0, \psi_0 + \alpha t, \Omega_1 t, \alpha)$$

with frequency vectors Ω_1 and $\alpha\Omega_2$.

Proof. An essential idea of the KAM method is to construct a simplifying transformation, consisting of infinitely many successive steps (referred to as KAM steps) of iterations, so that after each step the new perturbation terms of the transformed system are much smaller than the ones in the previous system. As all KAM steps can be carried out inductively, below, we only describe one step of KAM iteration in more detail.

Let Φ be an approximation function as defined in Definition 1.2. We choose $\tau > 0$, $0 < \mu < 1/4$, $\varpi \geq N_1 + N_2 + 2$ and T_0 sufficiently large such that

$$\Phi(T_0) < e^{\varpi}\mu^{\tau} \quad \text{and} \quad \int_{T_0}^{\infty} \log \frac{1}{\Phi(K)} \frac{dK}{K^2} < \frac{r_0}{4} \frac{\tau \log \frac{1}{\mu}}{(N_1 + N_2) \log \frac{4(N_1+N_2)}{\mu}}.$$

Following [16], we define the function

$$\Psi(T) := T^{-\varpi}\Phi(T), \quad T \geq 1$$

and put

$$T_\nu := \Psi^{-1}(\Psi(T_0)\mu^{\tau\nu}), \quad \nu \geq 0, \quad T_0 \geq 1.$$

Clearly, we have $\Psi \searrow 0$ as $T \to \infty$ and $1 \leq T_0 < T_i < T_j$ for $0 < i < j$ and fixed τ. Initially, we set $0 < \mu < \mu^*(< \frac{1}{4})$ for μ^* sufficiently small, and $\kappa > \tau$, $R_0 = R, Q_0 = Q$, $I_0 = I, \varphi_0 = \varphi$, $h_0 = h, r_0 = r > 0, 0 < s_0 = s < 1$. Define the sequences

$$\begin{cases} r_{\nu+1} = r_\nu - \frac{2(N_1+N_2)}{T_\nu} \log \frac{4(N_1+N_2)}{\mu}, \quad r_\nu = r_0 - 2(N_1 + N_2) \log \frac{4(N_1+N_2)}{\mu} \sum\limits_{\mu=0}^{\nu-1} \frac{1}{T_\mu}, \\ s_\nu = s_0 \mu^{\kappa\nu}, \\ h_\nu = \frac{\gamma}{2^\nu T_\nu} \Phi(T_\nu), \end{cases} \tag{2.7}$$

whence Lemma 13.2 in [16] implies that

$$\sum_{\nu=0}^{\infty} \frac{1}{T_\nu} \leq \frac{r_0}{4(N_1 + N_2) \log \frac{4(N_1+N_2)}{\mu}} \quad \text{and} \quad r_\nu \geq \frac{r_0}{2}. \tag{2.8}$$

We suppose that after ν steps, the transformed system is of the form

$$\begin{cases} \dot{I_\nu} = R_\nu(I_\nu, \varphi_\nu, t, a_\nu), \\ \dot{\varphi_\nu} = a_\nu + Q_\nu(I_\nu, \varphi_\nu, t, a_\nu), \end{cases} \tag*{$(2.9)_\nu$}$$

where the odd function R_ν and the even function Q_ν are real analytic on the domain $D_{r_\nu, s_\nu} \times O_{s_\nu}$ and have shell functions satisfying

$$|\widetilde{R_\nu}|_{r_\nu, s_\nu, h_\nu} < \gamma s_\nu^2, \quad |\widetilde{Q_\nu}|_{r_\nu, s_\nu, h_\nu} < \gamma s_\nu. \tag*{$(2.10)_\nu$}$$

We look for a change of variables S_ν defined in a smaller domain $D_{r_{\nu+1},s_{\nu+1}} \times O_{h_{\nu+1}}$, such that the system $(2.9)_\nu$ is transformed into the form $(2.9)_{\nu+1}$ and satisfies $(2.10)_{\nu+1}$. We construct the desired change of variables S_ν in the form

$$\begin{cases} I_\nu = I_{\nu+1} + f_{\nu 0}(\varphi_{\nu+1}, t, a_{\nu+1}) + f_{\nu 1}(\varphi_{\nu+1}, t, a_{\nu+1}) I_{\nu+1}, \\ \varphi_\nu = \varphi_{\nu+1} + g_{\nu 0}(\varphi_{\nu+1}, t, a_{\nu+1}), \\ a_\nu = a_{\nu+1} + \phi_\nu(a_{\nu+1}), \end{cases} \tag{2.11}$$

where

$$\begin{aligned} f_{\nu j}(\varphi_{\nu+1}, t, a_{\nu+1}) &= \sum_{|k| \le T_{\nu+1}} f^j_{k_1,k_2} e^{i\langle k_1, \Omega_1\rangle t + i\langle k_2, \Omega_2\rangle \varphi_{\nu+1}}, \quad j = 0, 1 && (2.12) \\ g_{\nu 0}(\varphi_{\nu+1}, t, a_{\nu+1}) &= \sum_{|k| \le T_{\nu+1}} g^0_{k_1,k_2} e^{i\langle k_1, \Omega_1\rangle t + i\langle k_2, \Omega_2\rangle \varphi_{\nu+1}}. && (2.13) \end{aligned}$$

In the actual construction we search for the inverse $\mathrm{id} - \sigma_\nu$ of $\mathrm{id} + \phi_\nu$, working with

$$a_\nu = a_{\nu+1} + \sigma_\nu(a_\nu).$$

It follows from $(2.9)_\nu$, $(2.9)_{\nu+1}$ and (2.11) that

$$\begin{aligned} &R_\nu(I_{\nu+1} + f_{\nu 0} + f_{\nu 1} I_{\nu+1}, \varphi_{\nu+1} + g_{\nu 0}, t, a_{\nu+1} + \sigma_\nu) \\ &\quad = R_{\nu+1} + \frac{\partial f_{\nu 0}}{\partial \varphi_{\nu+1}}(a_{\nu+1} + Q_{\nu+1}) + \frac{\partial f_{\nu 0}}{\partial t} + \frac{\partial f_{\nu 1}}{\partial \varphi_{\nu+1}}(a_{\nu+1} + Q_{\nu+1}) I_{\nu+1} \\ &\qquad + \frac{\partial f_{\nu 1}}{\partial t} I_{\nu+1} + f_{\nu 1} R_{\nu+1} \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &Q_\nu(I_{\nu+1} + f_{\nu 0} + f_{\nu 1} I_{\nu+1}, \varphi_{\nu+1} + g_{\nu 0}, t, a_{\nu+1} + \sigma_\nu) + a_{\nu+1} + \sigma_\nu \\ &\quad = a_{\nu+1} + Q_{\nu+1} + \frac{\partial g_{\nu 0}}{\partial t} + \frac{\partial g_{\nu 0}}{\partial \varphi_{\nu+1}}(a_{\nu+1} + Q_{\nu+1}). \end{aligned} \tag{2.15}$$

Using a Taylor expansion, we have

$$R_\nu(I_\nu, \varphi_\nu, t, a_\nu) = R_{\nu 0}(\varphi_\nu, t, a_\nu) + R_{\nu 1}(\varphi_\nu, t, a_\nu) I_\nu + O(I_\nu^2)$$

and

$$Q_\nu(I_\nu, \varphi_\nu, t, a_\nu) = Q_{\nu 0}(\varphi_\nu, t, a_\nu) + O(I_\nu).$$

For $R_{\nu 0}, R_{\nu 1}$ and $Q_{\nu 0}$ we have

$$\begin{aligned} R_{\nu j}(\varphi_\nu, t, a_\nu) &= \sum_{(k_1,k_2) \in \mathbb{Z}^{N_1} \times \mathbb{Z}^{N_2}} R^j_{k_1 k_2} e^{i\langle k_1, \Omega_1\rangle t + i\langle k_2, \Omega_2\rangle \varphi_\nu}, \quad j = 0, 1 \\ Q_{\nu 0}(\varphi_\nu, t, a_\nu) &= \sum_{(k_1,k_2) \in \mathbb{Z}^{N_1} \times \mathbb{Z}^{N_2}} Q^0_{k_1 k_2} e^{i\langle k_1, \Omega_1\rangle t + i\langle k_2, \Omega_2\rangle \varphi_\nu} \end{aligned}$$

and define the Fourier truncations

$$
\begin{aligned}
R_{\nu j}^{T_{\nu+1}}(\varphi_\nu, t, a_\nu) &= \sum_{|k|\le T_{\nu+1}} R_{k_1 k_2}^{j} e^{i\langle k_1,\Omega_1\rangle t + i\langle k_2,\Omega_2\rangle \varphi_\nu}, \quad j=0,1 \\
Q_{\nu 0}^{T_{\nu+1}}(\varphi_\nu, t, a_\nu) &= \sum_{|k|\le T_{\nu+1}} Q_{k_1 k_2}^{0} e^{i\langle k_1,\Omega_1\rangle t + i\langle k_2,\Omega_2\rangle \varphi_\nu}.
\end{aligned}
$$

Omitting all the smaller terms of order $O(I_\nu)$, the functions $f_{\nu 0}, f_{\nu 1}$ and $g_{\nu 0}$ in the transformation (2.11) are determined by the equations

$$
\frac{\partial f_{\nu j}}{\partial t} + \frac{\partial f_{\nu j}}{\partial \varphi_{\nu+1}} a_{\nu+1} = R_{\nu j}^{T_{\nu+1}}, \quad \frac{\partial g_{\nu 0}}{\partial t} + \frac{\partial g_{\nu 0}}{\partial \varphi_{\nu+1}} a_{\nu+1} = Q_{\nu 0}^{T_{\nu+1}} + \sigma_\nu. \tag{2.16}
$$

Since the functions $R_{\nu j}$ are odd, it follows that $\mathrm{av}(R_{\nu j}^{T_{\nu+1}}) = 0$. We take $\sigma_\nu = \mathrm{av}(-Q_{\nu 0}^{T_{\nu+1}})$. Hence, from (2.16) we get

$$
\begin{aligned}
i\left(\langle k_1,\Omega_1\rangle + \langle k_2, a_{\nu+1}\Omega_2\rangle\right) f_{k_1,k_2}^{j} &= R_{k_1,k_2}^{j}, \qquad j=1,2, \qquad 0<|k|\le T_{\nu+1}, \\
i\left(\langle k_1,\Omega_1\rangle + \langle k_2, a_{\nu+1}\Omega_2\rangle\right) g_{k_1,k_2}^{0} &= Q_{k_1,k_2}^{0}, \qquad\qquad\qquad\quad 0<|k|\le T_{\nu+1},
\end{aligned} \tag{2.17}
$$

and choose

$$
f_{0,0}^1 = 0, \quad f_{0,0}^2 = 0, \quad g_{0,0} = 0.
$$

We note that

$$
\begin{aligned}
|\langle k_1,\Omega_1\rangle + \langle k_2, a_{\nu+1}\Omega_2\rangle| &\ge |\langle k_1,\Omega_1\rangle + \langle k_2, \alpha\Omega_2\rangle| - |\langle k_2, (a_{\nu+1}-\alpha)\Omega_2\rangle| \\
&\ge \gamma\Phi(|k|) - T_{\nu+1}h_{\nu+1} > \gamma\Phi(T_{\nu+1}) - \frac{\gamma}{2^{\nu+1}}\Phi(T_{\nu+1}) \\
&> \frac{\gamma}{2}\Phi(T_{\nu+1})
\end{aligned}
$$

when $0<|k|\le T_{\nu+1}$ and $a_{\nu+1}\in A_\gamma + h_{\nu+1}$, for an appropriate $\alpha\in A_\gamma$. Hence, both equations in (2.16) are solvable. It also follows from (2.16) that the functions $f_{\nu j}$ are even and $g_{\nu 0}$ is odd. By (2.14) and (2.15), we see that the functions $R_{\nu+1}$ and $Q_{\nu+1}$ are odd and even, respectively. From (1.8) and (2.17) we have, for $0<|k|\le T_{\nu+1}$

$$
\begin{aligned}
|f_{k_1,k_2}^0| &\le \frac{2}{\gamma\Phi(T_{\nu+1})}|R_{k_1,k_2}^0| \le \frac{2}{\gamma\Phi(T_{\nu+1})}|R_{\nu 0}|_{r_\nu,s_\nu,h_\nu} e^{-|k|r_\nu} \\
&\le \frac{2}{\gamma\Phi(T_{\nu+1})}|R_\nu|_{r_\nu,s_\nu,h_\nu} e^{-|k|r_\nu}, \\
|f_{k_1,k_2}^1| &\le \frac{2}{\gamma\Phi(T_{\nu+1})}|R_{k_1,k_2}^1| \le \frac{2}{\gamma\Phi(T_{\nu+1})}|R_{\nu 1}|_{r_\nu,s_\nu,h_\nu} e^{-|k|r_\nu} \\
&\le \frac{2}{\gamma s_\nu\Phi(T_{\nu+1})}|R_\nu|_{r_\nu,s_\nu,h_\nu} e^{-|k|r_\nu}, \\
|g_{k_1,k_2}^0| &\le \frac{2}{\gamma\Phi(T_{\nu+1})}|Q_{k_1,k_2}^0| \le \frac{2}{\gamma\Phi(T_{\nu+1})}|Q_{\nu 0}|_{r_\nu,s_\nu,h_\nu} e^{-|k|r_\nu} \\
&\le \frac{2}{\gamma\Phi(T_{\nu+1})}|Q_\nu|_{r_\nu,s_\nu,h_\nu} e^{-|k|r_\nu}.
\end{aligned}
$$

Hence, it follows from (2.12) and (2.13) that

$$
\begin{aligned}
|I_\nu - I_{\nu+1}| &= |f_{\nu 0}(\varphi_{\nu+1}, t, a_{\nu+1})| + |f_{\nu 1}(\varphi_{\nu+1}, t, a_{\nu+1})||I_{\nu+1}| \\
&\le \sum_{|k|\le T_{\nu+1}} |f^0_{k_1,k_2}|_{r_\nu,s_\nu,h_\nu} e^{-|k|(r_\nu - r_{\nu+1})} + s_{\nu+1} \sum_{|k|\le T_{\nu+1}} |f^1_{k_1,k_2}|_{r_\nu,s_\nu,h_\nu} e^{-|k|(r_\nu - r_{\nu+1})} \\
&\le \sum_{|k|\le T_{\nu+1}} \frac{2}{\gamma\Phi(T_{\nu+1})} |R_\nu|_{r_\nu,s_\nu,h_\nu} e^{-|k|(r_\nu - r_{\nu+1})} + s_{\nu+1} \sum_{|k|\le T_{\nu+1}} \frac{2}{\gamma s_\nu \Phi(T_{\nu+1})} |R_\nu|_{r_\nu,s_\nu,h_\nu} e^{-|k|(r_\nu - r_{\nu+1})} \\
&\le \frac{2s_\nu(s_\nu + s_{\nu+1})}{\Phi(T_{\nu+1})} (N_1+N_2)! 4^{N_1+N_2} \left(\frac{2(N_1+N_2)}{T_{\nu+1}} \log \frac{4(N_1+N_2)}{\mu} \right)^{-(N_1+N_2)} \qquad (2.18) \\
&\le \frac{2s_\nu(s_\nu+s_{\nu+1})(N_1+N_2)! 4^{N_1+N_2} T_{\nu+1}^{N_1+N_2}}{T_{\nu+1}^{\varpi} \Psi(T_{\nu+1})} \le 2c_1 s^2 \mu^{(2\kappa-\tau)\nu-\tau} \le 2c_1 s^2 \mu^{\kappa\nu-\tau} \\
&\le 2c_1 s \left[\frac{1}{2c_1}(1-\mu^\kappa)\mu^\kappa \right] \mu^{\kappa\nu-\tau} = s\mu^{\kappa\nu} - s\mu^{\kappa(\nu+1)} = s_\nu - s_{\nu+1}
\end{aligned}
$$

and

$$
\begin{aligned}
|\varphi_\nu - \varphi_{\nu+1}| &= |g_{\nu 0}(\varphi_{\nu+1}, t, a_{\nu+1})| \le \sum_{|k|\le T_{\nu+1}} |g^0_{k_1,k_2}|_{r_\nu,s_\nu,h_\nu} e^{-|k|(r_\nu - r_{\nu+1})} \\
&\le \sum_{|k|\le T_{\nu+1}} \frac{2}{\gamma\Phi(T_{\nu+1})} |Q_\nu|_{r_\nu,s_\nu,h_\nu} e^{-|k|(r_\nu - r_{\nu+1})} \qquad (2.19) \\
&\le \frac{2}{\gamma\Phi(T_{\nu+1})} |Q_\nu|_{r_\nu,s_\nu,h_\nu} (N_1+N_2)! 4^{N_1+N_2} \left(\frac{2(N_1+N_2)}{T_{\nu+1}} \log \frac{4(N_1+N_2)}{\mu} \right)^{-(N_1+N_2)} \\
&\le \frac{2(N_1+N_2)! 4^{N_1+N_2} s_\nu T_{\nu+1}^{N_1+N_2}}{T_{\nu+1}^{\varpi} \Psi(T_{\nu+1})} \le \frac{c_1 s_\nu}{T_{\nu+1}^{\varpi - N_1 - N_2} \mu^{\tau(\nu+1)}} \\
&\le \frac{c_1 s \mu^{\kappa-\tau}}{T_{\nu+1}\mu^\tau} \le \frac{c_1 s}{T_\nu \mu^\tau} \le \frac{1}{\omega T_\nu}
\end{aligned}
$$

provided that

$$0 < s < \min\left\{ \frac{1}{2c_1}(\mu^\kappa - \mu^{2\kappa}); \frac{\mu^\tau}{c_1\omega} \right\}$$

where

$$c_1 := \frac{2(N_1+N_2)! 4^{N_1+N_2}}{\Psi(T_0)}, \quad \omega := \max\{|\omega_1|, \ldots, |\omega_{N_1}|, |\omega_{N_1+1}|, \ldots, |\omega_{N_1+N_2}|\}.$$

Hence, we obtain

$$|I_\nu| \le |I_{\nu+1}| + |I_\nu - I_{\nu+1}| \le s_{\nu+1} + s_\nu - s_{\nu+1} = s_\nu$$

and

$$|\operatorname{Im} \widetilde{\theta}_{j\nu}| \;=\; |\operatorname{Im} \omega_j \varphi_\nu| \le |\operatorname{Im} \omega_j \varphi_{\nu+1}| + |\operatorname{Im} \omega_j \varphi_\nu - \operatorname{Im} \omega_j \varphi_{\nu+1}|$$

$$\begin{aligned}&\le |\operatorname{Im}\omega_j\varphi_{\nu+1}|+|\omega_j\varphi_\nu-\omega_j\varphi_{\nu+1}|\le r_{\nu+1}+\frac{1}{T_\nu}\\&\le r_{\nu+1}+\frac{2(N_1+N_2)}{T_\nu}\log\frac{4(N_1+N_2)}{\mu}=r_\nu.\end{aligned}$$

This implies that the transformation $\mathcal{S}_\nu$ defined by (2.11) satisfies $\mathcal{S}_\nu(D_{r_{\nu+1},s_{\nu+1}})\subset D_{r_\nu,s_\nu}$. Note that

$$\begin{aligned}r_\nu-r_{\nu+1}&=\frac{2(N_1+N_2)}{T_\nu}\log\frac{4(N_1+N_2)}{\mu}\ge\frac{(N_1+N_2)}{T_\nu}\log\frac{4(N_1+N_2)}{\mu}\\&\ge\frac{2}{T_\nu}\ge\frac{2}{T_{\nu+1}},\end{aligned}$$

it follows from (2.18), (2.19), (2.11) and the second inequality of (1.8) that

$$\begin{aligned}|\mathcal{S}_\nu-id|&\le|f_{\nu0}(\varphi_{\nu+1},t,a_{\nu+1})|+|f_{\nu1}(\varphi_{\nu+1},t,a_{\nu+1})|\cdot|I_{\nu+1}|+|g_{\nu0}(\varphi_{\nu+1},t,a_{\nu+1})|\\&\le 2c_1s^2\mu^{\kappa\nu-\tau}+\frac{c_1s\mu^{(\kappa-\tau)\nu}}{T_{\nu+1}\mu^\tau}\le 3c_1s\mu^{(\kappa-\tau)\nu-\tau}\le c_2\mu^{(\kappa-\tau)\nu}\end{aligned}\tag{2.20}$$

and

$$\begin{aligned}|D\mathcal{S}_\nu-Id|&\le|f_{\nu1}|+\frac{1}{r_\nu-r_{\nu+1}}(|g_{\nu0}|+|f_{\nu0}|+|f_{\nu1}||I_{\nu+1}|)\\&\le\frac{2s_\nu(N_1+N_2)!4^{N_1+N_2}T_{\nu+1}^{N_1+N_2}}{T_{\nu+1}^{\varpi}\Psi(T_{\nu+1})}\\&\quad+\frac{1}{r_\nu-r_{\nu+1}}\left[\frac{c_1s\mu^{(\kappa-\tau)\nu}}{T_{\nu+1}\mu^\tau}+\frac{2s_\nu(s_\nu+s_{\nu+1})(N_1+N_2)!4^{N_1+N_2}T_{\nu+1}^{N_1+N_2}}{T_{\nu+1}^{\varpi}\Psi(T_{\nu+1})}\right]\\&\le\frac{c_1s\mu^{(\kappa-\tau)\nu}}{\mu^\tau}+\frac{c_1s\mu^{(\kappa-\tau)\nu}}{\mu^\tau}+\frac{2c_1s\mu^{(\kappa-\tau)\nu}}{\mu^\tau}\qquad(2.21)\\&\le\frac{4c_1s\mu^{(\kappa-\tau)\nu}}{\mu^\tau}\le c_2\mu^{(\kappa-\tau)\nu},\end{aligned}$$

where $c_2:=\frac{4c_1}{\mu^\tau}$. Moreover, by the definition of σ_ν we have

$$|\sigma_\nu|=|\operatorname{av}(-Q_{\nu0}^{T_{\nu+1}})|=|Q_{00}^0|\le|Q_{00}|_{r_0,s_0,h_0}e^{-|k|r_0}\le|Q_0|_{r_0,s_0,h_0}<\gamma s<\frac{h}{4}$$

provided that $0<s<\frac{h}{4\gamma}$. Thus, for the mapping $a_\nu\mapsto a_{\nu+1}=a_\nu-\sigma_\nu(a_\nu)$, by Lemma A.3 of Appendix A in [15], there exists a real analytic inverse mapping

$$\mathrm{id}+\phi:a_{\nu+1}\mapsto a_\nu$$

such that $(\mathrm{id}+\phi)(O_{h_{\nu+1}})\subset O_{h_\nu}$ with the estimates

$$|\phi|,\frac{h}{4}|D\phi|\le\gamma s$$

on $O_{h/4}$. We therefore have obtained a transformation $\widetilde{\mathcal{F}}_\nu = (S_\nu, \mathrm{id} + \phi)$ such that

$$\widetilde{\mathcal{F}}_\nu(D_{r_{\nu+1},s_{\nu+1}} \times O_{h_{\nu+1}}) \subset D_{r_\nu,s_\nu} \times O_{h_\nu}. \tag{2.22}$$

Now, we estimate the functions $R_{\nu+1}$ and $Q_{\nu+1}$. From (2.14), (2.15) and (2.16), we have

$$R_{\nu+1} = (1 + f_{\nu1})^{-1} \left[M_1 + M_2 + (M_3 + M_4) I_\nu + M_5\right] \tag{2.23}$$

and

$$Q_{\nu+1} = \left(1 + \frac{\partial g_{\nu0}}{\partial \varphi_{\nu+1}}\right)^{-1} (M_6 + M_7), \tag{2.24}$$

where

$$\begin{aligned}
M_1 &= R_{\nu0}(\varphi_{\nu+1} + g_{\nu0}, t, a_{\nu+1} + \sigma_\nu) - R_{\nu0}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu), \\
M_2 &= R_{\nu0}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu) - R_{\nu0}^{T_{\nu+1}}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu), \\
M_3 &= R_{\nu1}(\varphi_{\nu+1} + g_{\nu0}, t, a_{\nu+1} + \sigma_\nu) - R_{\nu1}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu), \\
M_4 &= R_{\nu1}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu) - R_{\nu1}^{T_{\nu+1}}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu), \\
M_5 &= R_{\nu1}^{T_{\nu+1}}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu)\,(f_{\nu0} + f_{\nu1} I_{\nu+1}) - \left(\frac{\partial f_{\nu0}}{\partial \varphi_{\nu+1}} + \frac{\partial f_{\nu1}}{\partial \varphi_{\nu+1}} I_{\nu+1}\right) Q_{\nu+1}, \\
M_6 &= Q_{\nu0}(\varphi_{\nu+1} + g_{\nu0}, t, a_{\nu+1} + \sigma_\nu) - Q_{\nu0}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu), \\
M_7 &= Q_{\nu0}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu) - Q_{\nu0}^{T_{\nu+1}}(\varphi_{\nu+1}, t, a_{\nu+1} + \sigma_\nu).
\end{aligned}$$

First, we have the following estimates

$$\left|\frac{\partial g_{\nu0}}{\partial \varphi_{\nu+1}}\right| \le \frac{1}{r_\nu - r_{\nu+1}} |g_{\nu0}|_{r_\nu,s_\nu,h_\nu} \le \frac{T_\nu}{2} \cdot \frac{c_1 s}{T_\nu \mu^\tau} \le \frac{1}{4}$$

provided that $0 < s < \frac{2}{c_2}$, and

$$\begin{aligned}
|M_6| &\le \left|\frac{\partial Q_{\nu0}}{\partial \varphi_{\nu+1}}\right| |g_{\nu0}| \le \frac{1}{r_\nu - r_{\nu+1}} |Q_{\nu0}|_{r_\nu,s_\nu,h_\nu} |g_{\nu0}| \\
&\le \frac{1}{r_\nu - r_{\nu+1}} |Q_\nu|_{r_\nu,s_\nu,h_\nu} |g_{\nu0}| \le \frac{T_\nu}{2} \gamma s_\nu \frac{c_1 s}{T_\nu \mu^\tau} = c_1 \gamma s s_\nu \mu^{-\tau}.
\end{aligned}$$

Note that $T_\nu \to \infty$ as $\mu \to 0$, we can choose $\mu^* > 0$ so small that $T_{\nu+1}^{N_1+N_2} e^{-\frac{r_0}{2} T_{\nu+1}} < s$ as $0 < \mu < \mu^*$. Thus, from Lemma A.2 in [15] we have

$$\begin{aligned}
|M_7| &\le \sum_{|k| > T\nu+1} |Q^0_{k_1k_2} e^{i\langle k_1,\Omega_1\rangle t + i\langle k_2,\Omega_2\rangle \varphi_{\nu+1}}| \le c T_{\nu+1}^{N_1+N_2} e^{-\frac{r_0}{2} T_{\nu+1}} |Q_{\nu0}|_{r_\nu,s_\nu,h_\nu} \\
&\le c T_{\nu+1}^{N_1+N_2} e^{-\frac{r_0}{2} T_{\nu+1}} |Q_\nu|_{r_\nu,s_\nu,h_\nu} \le c \gamma s s_\nu,
\end{aligned}$$

where c only depends on $N_1 + N_2$. It follows that

$$\begin{aligned}
|Q_{\nu+1}|_{r_{\nu+1},s_{\nu+1},h_{\nu+1}} &\le \frac{1}{1 - |\frac{\partial g_{\nu0}}{\partial \varphi_{\nu+1}}|} (|M_6| + |M_7|) \le \frac{1}{1 - \frac{1}{4}} (c_1 \gamma s s_\nu \mu^{-\tau} + c \gamma s s_\nu) \\
&< \gamma s_{\nu+1}
\end{aligned}$$

provided that $0 < s < \frac{3\mu^\kappa}{c_2+4c}$. Moreover, we also have the estimates

$$\begin{aligned}
|M_5| &\leq \left|\sum_{|k|\leq T\nu+1} R^1_{k_1k_2} e^{i\langle k_1,\Omega_1\rangle t+i\langle k_2,\Omega_2\rangle\varphi_{\nu+1}}\right| |f_{\nu 0}+f_{\nu 1}I_{\nu+1}| + \left|\left(\frac{\partial f_{\nu 0}}{\partial\varphi_{\nu+1}} + \frac{\partial f_{\nu 1}}{\partial\varphi_{\nu+1}} I_{\nu+1}\right) Q_{\nu+1}\right| \\
&\leq \left(\sum_{|k|\leq T\nu+1} \frac{1}{s_\nu}|R_\nu|_{r_\nu,s_\nu,h_\nu} e^{-|k|(r_\nu-r_{\nu+1})}\right) |f_{\nu 0}+f_{\nu 1}I_{\nu+1}| + \frac{1}{r_\nu-r_{\nu+1}}|f_{\nu 0}+f_{\nu 1}I_{\nu+1}||Q_{\nu+1}| \\
&\leq \frac{c_1\gamma s_\nu}{\mu^{\tau(\nu+1)}}\cdot\frac{c_1 s_\nu(s_\nu+s_{\nu+1})}{\mu^{\tau(\nu+1)}} + \frac{\gamma s_{\nu+1}s_\nu(s_\nu+s_{\nu+1})(N_1+N_2)!4^{N_1+N_2}T_{\nu+1}^{N_1+N_2+1}}{T^{\varpi}_{\nu+1}\Psi(T_{\nu+1})} \\
&\leq \frac{\gamma c_1^2 s_\nu^2(s_\nu+s_{\nu+1})}{\mu^{2\tau(\nu+1)}} + \frac{\gamma c_1 s_{\nu+1}s_\nu(s_\nu+s_{\nu+1})}{\mu^{\tau(\nu+1)}},
\end{aligned}$$

and

$$\begin{aligned}
|M_1| &\leq \left|\frac{\partial R_{\nu 0}}{\partial\varphi_{\nu+1}}\right||g_{\nu 0}| \leq \frac{1}{r_\nu-r_{\nu+1}}|R_{\nu 0}|_{r_\nu,s_\nu,h_\nu}|g_{\nu 0}| \leq \frac{1}{r_\nu-r_{\nu+1}}|R_\nu|_{r_\nu,s_\nu,h_\nu}|g_{\nu 0}| \\
&\leq \frac{T_\nu}{2}\gamma s_\nu^2\frac{c_1 s}{T_\nu\mu^\tau} = c_1\gamma s s_\nu^2\mu^{-\tau}, \\
|M_2| &\leq \sum_{|k|>T\nu+1}|R^0_{k_1k_2}e^{i\langle k_1,\Omega_1\rangle t+i\langle k_2,\Omega_2\rangle\varphi_{\nu+1}}| \leq cT_{\nu+1}^{N_1+N_2}e^{-\frac{r_0}{2}T_{\nu+1}}|R_{\nu 0}|_{r_\nu,s_\nu,h_\nu} \\
&\leq cT_{\nu+1}^{N_1+N_2}e^{-\frac{r_0}{2}T_{\nu+1}}|R_\nu|_{r_\nu,s_\nu,h_\nu} \leq c\gamma s s_\nu^2, \\
|M_3| &\leq \left|\frac{\partial R_{\nu 1}}{\partial\varphi_{\nu+1}}\right||g_{\nu 0}| \leq \frac{1}{r_\nu-r_{\nu+1}}|R_{\nu 1}|_{r_\nu,s_\nu,h_\nu}|g_{\nu 0}| \\
&\leq \frac{1}{(r_\nu-r_{\nu+1})s_\nu}|R_\nu|_{r_\nu,s_\nu,h_\nu}|g_{\nu 0}| = c_1\gamma s s_\nu\mu^{-\tau}, \\
|M_4| &\leq \sum_{|k|>T\nu+1}|R^1_{k_1k_2}e^{i\langle k_1,\Omega_1\rangle t+i\langle k_2,\Omega_2\rangle\varphi_{\nu+1}}| \leq cT_{\nu+1}^{N_1+N_2}e^{-\frac{r_0}{2}T_{\nu+1}}|R_{\nu 1}|_{r_\nu,s_\nu,h_\nu} \\
&\leq cT_{\nu+1}^{N_1+N_2}e^{-\frac{r_0}{2}T_{\nu+1}}\frac{1}{s_\nu}|R_\nu|_{r_\nu,s_\nu,h_\nu} \leq c\gamma s s_\nu.
\end{aligned}$$

Note that

$$|f_{\nu 1}| \leq \sum_{|k|\leq T_{\nu+1}}\frac{2}{\gamma s_\nu\Phi(T_{\nu+1})}|R_\nu|_{r_\nu,s_\nu,h_\nu}e^{-|k|(r_\nu-r_{\nu+1})} \leq \frac{2c_1 s}{\mu^\tau} < 1,$$

whence we obtain the estimate

$$\begin{aligned}
&|R_{\nu+1}|_{r_{\nu+1},s_{\nu+1},h_{\nu+1}} \\
&\quad\leq \frac{1}{1-\frac{2c_1 s}{\mu^\tau}}\Big[c_1\gamma s s_\nu^2\mu^{-\tau} + c\gamma s s_\nu^2 + \left(c_1\gamma s s_\nu\mu^{-\tau}+c\gamma s s_\nu\right)s_\nu \\
&\qquad + \frac{\gamma c_1^2 s_\nu^2(s_\nu+s_{\nu+1})}{\mu^{2\tau(\nu+1)}} + \frac{\gamma c_1 s_{\nu+1}s_\nu(s_\nu+s_{\nu+1})}{\mu^{\tau(\nu+1)}}\Big] \\
&\quad\leq \frac{2\mu^\tau}{\mu^\tau-2c_1 s}\left(c_1\mu^{-\tau}+c+c_1^2\mu^{-2\tau}+c_1\right)s\gamma s_\nu^2 \leq \gamma s_{\nu+1}^2,
\end{aligned}$$

provided that

$$0 < s < \frac{\mu^{2(\kappa+\tau)}}{2(c_1\mu^\tau + c\mu^{2\tau} + c_1^2 + c_1\mu^{2\tau} + c_1\mu^{2\kappa+\tau})}.$$

By induction, we can prove that for any $\nu \geq 0$ there is a sequence $\widetilde{\mathcal{F}}_\nu = (\mathcal{S}_\nu, \mathrm{id} + \phi)$ of transformations such that (2.22) and $(2.10)_{\nu+1}$ are fulfilled if

$$0 < s < \min\left\{ \frac{1}{2c_1}(\mu^\kappa - \mu^{2\kappa}); \frac{\mu^\tau}{c_1\omega}; \frac{h}{4\gamma}; \frac{3\mu^\kappa}{c_2+4c}; \frac{\mu^{2(\kappa+\tau)}}{2(c_1\mu^\tau + c\mu^{2\tau} + c_1^2 + c_1\mu^{2\tau} + c_1\mu^{2\kappa+\tau})} \right\}.$$

It remains to establish the uniform convergence of the sequence

$$\mathcal{S}^\nu := \mathcal{S}_0 \circ \mathcal{S}_1 \circ \ldots \circ \mathcal{S}_\nu \ : \ D_{r_{\nu+1},s_{\nu+1}} \times O_{h_{\nu+1}} \longrightarrow D_{r,s} \times O_h.$$

It follows from (2.21) that

$$|D\mathcal{S}_\chi(\mathcal{S}_{\chi+1} \circ \cdots \circ \mathcal{S}_\nu)| \leq 1 + c_2\mu^{(\kappa-\tau)\chi}, \quad \chi = 0, 1, \ldots, \nu-1.$$

We infer

$$|D\mathcal{S}^{\nu-1}| \leq (1+c_2)\cdots\left(1 + c_2\mu^{(\kappa-\tau)(\nu-1)}\right) \leq e^{c_2(1+\mu^{(\kappa-\tau)}+\cdots+\mu^{(\kappa-\tau)(\nu-1)})} \leq e^{2c_2}$$

and hence

$$|\mathcal{S}^\nu - \mathcal{S}^{\nu-1}| \leq |\mathcal{S}^{\nu-1}(\mathcal{S}_\nu) - \mathcal{S}^{\nu-1}| \leq e^{2c_2}|\mathcal{S}_\nu - id| \leq 2c_2e^{2c_2}\mu^{(\kappa-\tau)\nu}.$$

The same inequality holds for $\nu = 0$ if we define $\mathcal{S}^{-1} := \mathrm{id}$. This implies that the sequence

$$\mathcal{S}^\nu = \mathcal{S}^0 + \sum_{i=0}^{\nu}(\mathcal{S}^i - \mathcal{S}^{i-1})$$

converges uniformly on $D_{\frac{r_0}{2},0} \times A_\gamma$. This domain has zero width in the I-direction, whence only φ, t and a can vary (the latter only within the Cantor set A_γ). To extend the limit function $\mathcal{S}^\infty$ to a valid transformation we use that the $\mathcal{S}_\nu$ are defined in (2.11) as affine transformations in I, linear polynomials with coefficients depending on φ, t and a. This makes the compositions $\mathcal{S}^\nu$ affine in I as well, and we just showed that the constant coefficients converge to $\mathcal{S}^\infty$. The linear coefficients are given by the derivatives

$$\mathcal{T}^\nu(\varphi_{\nu+1}, t, a_{\nu+1}) = \frac{\partial \mathcal{S}^\nu}{\partial I_{\nu+1}}(0, \varphi_{\nu+1}, t, a_{\nu+1})$$

at $I_{\nu+1} = 0$ and to show convergence we estimate how much these differ from the identity in $I_{\nu+1}$. The above considerations applied to $\mathcal{T}^\nu$ yield again the desired convergence; here we use that the second derivative with respect to $I_{\nu+1}$ vanishes. Now put

$$\mathcal{S}_\infty(I, \varphi, t, a) := \mathcal{S}^\infty(\varphi, t, a) + \mathcal{T}^\infty(\varphi, t, a)I$$

whence $\mathcal{S}_\infty$ is the desired transformation (2.4) and (2.6) follows from $(2.10)_\nu$. This completes the proof. $\square$

Remark 2.2 The Bruno condition (1.4) is weaker than a Diophantine condition with the same $\gamma > 0$ and it follows for $N_2 = 1$ similarly to Lemma 2 in [2] that for fixed $\epsilon > 0$ the measure of

$$A_{\epsilon L} = \left\{ \alpha \in \epsilon J : |\langle k_1, \Omega_1 \rangle + \langle k_2, \alpha\Omega_2 \rangle| \geq \epsilon L \Phi(|k|) \text{ for all } 0 \neq k \in \mathbb{Z}^{N_1+N_2} \right\} \tag{2.25}$$

where our result does not apply tends to ϵ as $L \downarrow 0$, with $\Phi(t)$ an approximation function as in Definition 1.2 and $J \subset (0, +\infty)$ a unit interval. This extends to $N_2 \geq 2$.

2.2 Hamiltonian systems

Consider the Hamiltonian system

$$\dot{I} = -\frac{\partial H(I, \varphi, t, a)}{\partial \varphi}, \quad \dot{\varphi} = a + \frac{\partial H(I, \varphi, t, a)}{\partial I} \tag{2.26}$$

on $\mathbb{R} \times \mathbb{T}$ with a one-dimensional parameter a. We assume that for fixed I and a we have $H(I, \cdot, \cdot, a) \in Q_r(\Omega_1, \Omega_2)$ with $\Omega_1 = (\omega_1, \dots, \omega_{N_1}) \in \mathbb{R}^{N_1}$ and $\Omega_2 = \omega_{N_1+1} \in \mathbb{R}$. The corresponding shell function $\widetilde{H}(I, \varphi, \theta, a)$ is real analytic in $D_{r,s} \times O_h$ and satisfies

$$|\widetilde{H}|_{r,s,h} < \gamma s^2. \tag{2.27}$$

Furthermore we assume that $H = H(I, \varphi, t, a)$ has for $I = 0$ zero average with respect to φ and t.

Theorem 2.3 *Under the above assumptions and if the Bruno condition (1.4) holds, there is an s^* such that for any positive $s < s^*$ there is a function $a_0 : A_\gamma \longrightarrow \mathbb{R}$ satisfying $|a_0(\alpha) - \alpha| < h$ for all $\alpha \in A_\gamma$ and a canonical change of variables*

$$I = \varrho + v(\varrho, \psi, \theta, \alpha), \quad \varphi = \psi + u(\psi, \theta, \alpha), \quad \alpha \in A_\gamma \tag{2.28}$$

which transforms system (2.26) with $a = a_0(\alpha)$ into a system

$$\dot{\varrho} = -\frac{\partial K(\varrho, \psi, t, \alpha)}{\partial \psi}, \quad \dot{\psi} = \alpha + \frac{\partial K(\varrho, \psi, t, \alpha)}{\partial \varrho} \tag{2.29}$$

such that

$$\frac{\partial K(0, \psi, t, \alpha)}{\partial \psi} = \frac{\partial K(0, \psi, t, \alpha)}{\partial \varrho} = \frac{\partial^2 K(0, \psi, t, \alpha)}{\partial \varrho \, \partial \psi} = 0. \tag{2.30}$$

It follows that for each ψ_0 the system (2.26) admits the quasi-periodic solution

$$I(t) = v(0, \psi_0 + \alpha t, \Omega_1 t, \alpha), \quad \varphi(t) = \psi_0 + \alpha t + u(0, \psi_0 + \alpha t, \Omega_1 t, \alpha)$$

with frequency vectors Ω_1 and $\alpha\Omega_2$.

Proof. We use the shell function $\widetilde{H}$ to define a Hamiltonian system in $N_1 + 1$ degrees of freedom. To this end we introduce the variables $E_1, \dots, E_{N_1}$ canonically conjugate to the angles $\theta_1, \dots, \theta_{N_1}$ and define

$$\widehat{H}(I, E, \varphi, \theta, a) := aI + \langle \Omega_1, E \rangle + \widetilde{H}(I, \varphi, \theta, a)$$

with resulting equations of motion

$$\begin{aligned}
\dot{I} &= -\frac{\partial \widetilde{H}(I,\varphi,\theta,a)}{\partial \varphi}, \quad \dot{E} \;=\; -\nabla_\theta \widetilde{H}(I,\varphi,\theta,a),\\
\dot{\varphi} &= a + \frac{\partial \widetilde{H}(I,\varphi,\theta,a)}{\partial I}, \quad \dot{\theta} \;=\; \Omega_1.
\end{aligned}$$

To this situation we apply the Main Theorem 1.7 of [16] with unperturbed part $aI + \langle \Omega_1, E\rangle + \mathrm{av}(\widetilde{H})(I,a)$ and perturbation $\widetilde{H}(I,\varphi,\theta,a) - \mathrm{av}(\widetilde{H})(I,a)$. Since the latter does not depend on E the ensuing transformation

$$(I,E,\varphi,\theta,a) \mapsto (\varrho,F,\psi,\theta,\alpha)$$

keeps θ fixed and yields

$$\widehat{K}(\varrho,F,\psi,\theta,\alpha) = \alpha\varrho + \langle \Omega_1, F\rangle + \widetilde{K}(\varrho,\psi,\theta,\alpha).$$

The resulting equations of motion are again independent of F and inserting $\theta = \Omega_1 t$ leads to the desired system (2.29). □

Remark 2.4 For $\Omega_2 \in \mathbb{R}^{N_2}$ with $N_2 \geq 2$ one can similarly define a Hamiltonian system in $N_1 + N_2$ degrees of freedom, with N_2 additional angular variables $\widetilde{\theta}$ to allow for $\widetilde{\theta} = \Omega_2\varphi$ and conjugate actions $J \in \mathbb{R}^{N_2}$.

3 Applications

In this section we apply Theorems 2.1 and 2.3 to non-autonomous vector fields in the plane. Specifically, we are interested in dynamical stability of the solution $(x(t), y(t)) \equiv (0,0)$ that stays at the origin for all times. The main idea is as follows: the families of quasi-periodic solutions constructed in the previous section determine invariant cylinders that are boundaries of tubular neighbourhoods around the t-axis. Such cylinders confine integral curves, and so the stability follows. We generalize results that have been obtained in [7, 8, 2, 9], where the cases of reversible and Hamiltonian systems were treated separately but with very similar proofs. Using the Lie-algebraic approach of [10, 4] we give a unified proof, bringing out more clearly the intimate relation between these results. In order to equally allow for a unified formulation we speak of a *conservative* vector field / system if a vector field / system is reversible *or* Hamiltonian; the conserved structure being the involution

$$(x,y,t) \mapsto (-x,y,-t)$$

or the area element (symplectic structure)

$$dx \wedge dy,$$

respectively.

3.1 Formulation of results

Consider the real analytic conservative system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = X_H(x,y,t) + Y(x,y,t) + Z(x,y,t) \tag{3.1}$$

where the quasi-periodic vector field on the right hand side is the sum of the Hamiltonian vector field X_H with Hamiltonian

$$H(x,y,t) = \frac{a(t)}{2m+2}y^{2m+2} + \frac{b(t)}{2n+2}x^{2n+2}, \quad m,n \in \mathbb{N}_0 \text{ with } m+n \geq 1, \tag{3.2}$$

a conservative vector field Y that has zero average and a conservative vector field Z consisting of higher order terms.

The coefficient functions a and b in (3.2) are quasi-periodic functions in t with basic frequency Ω_1. We remark that the vector field X_H is reversible if and only if a and b are even functions in t. The average of X_H coincides with the Hamiltonian vector field defined by $\mathrm{av}(H)$ and has equations of motion

$$\begin{cases} \dot{x} &= \quad \mathrm{av}(a)y^{2m+1} \\ \\ \dot{y} &= -\mathrm{av}(b)x^{2n+1}. \end{cases} \tag{3.3}$$

We require the coefficients $\mathrm{av}(a)$ and $\mathrm{av}(b)$ to be non-zero. Introducing the weights $\alpha := \frac{m+1}{m+n+2}$ and $\beta := \frac{n+1}{m+n+2} = 1-\alpha$ makes (3.2) a non-degenerate quasi-homogeneous function of degree $\delta := 2(n+1)\alpha = 2(m+1)\beta > 1$ (as we assumed $m+n \geq 1$ in (3.2)).

For $m = n = 0$ the system (3.3) is the harmonic oscillator with frequency $\sqrt{\mathrm{av}(a)\cdot\mathrm{av}(b)}$ if both coefficients have the same sign and a hyperbolic linear system if $\mathrm{av}(a)\cdot\mathrm{av}(b) < 0$. While our results remain true in this case this requires a slightly different proof and since this is well-known we prefer not to burden our presentation by trying to give case-dependent formulas for the transformations we use, but restrict from the outset to the case $m+n \geq 1$. Where helpful we comment on the case $m = n = 0$ as well.

We use the quasi-homogeneous gradation/filtration defined by the weight (α, β) to identify which terms are of higher order. Following [1] we say that a vector field is of order d if differentiation in the direction of the field raises the degree of any function by not less than d. Thus X_H is of order $\delta - 1$ and we require Z in (3.1) to have order strictly larger than $\delta - 1$.

In case the conservative vector field Z is Hamiltonian this means that all monomials $x^k y^l$ in the Hamiltonian function of Z have degree $\alpha k + \beta l > \delta$. In the reversible case this means that the monomials $x^i y^j$ of the x-component of Z have degree $\alpha i + \beta j > \delta - \beta$ and those of the y-component of Z to have degree strictly larger than $\delta - \alpha$. The weaker property $\mathcal{P}(m+1, n+1)$ in [9] requires the monomials of both components to all have degree $\alpha i + \beta j > \max(\delta - \alpha, \delta - \beta)$.

The quasi-periodic conservative vector field Y in (3.1) is proof-generated. Part of the procedure to obtain information on (3.1) is to transform away the time-dependent part of X_H and this procedure turns out to work for more general terms. Hence, the main requirement on Y is that $\mathrm{av}(Y) = 0$. Furthermore Y has to be of order strictly larger than $\sigma := \frac{1}{2}(\delta - 1)$. Thus, an order a bit less than the order $\delta - 1$ of the 'dominant' term X_H is allowed. With these definitions at hand we can formulate our result as follows.

Theorem 3.1 *Let $X_H + Y + Z$ be a quasi-periodic conservative vector field with Hamiltonian of X_H given by* (3.2)*, a conservative vector field Y with zero average of order strictly larger than σ and a conservative vector field Z of order strictly larger than $\delta - 1$. Assume that* $\mathrm{av}(a) \cdot \mathrm{av}(b) \neq 0$ *and the common frequency vector $\Omega = \Omega_1$ of a, b, Y and Z satisfies the Bruno condition* (1.4)*. Then the trivial solution $(x(t), y(t)) \equiv (0,0)$ of the system* (3.1) *is stable if and only if* $\mathrm{av}(a) \cdot \mathrm{av}(b) > 0$.

The proof constitutes the remaining §§ 3.2–3.4 below. For the benefit of the reader we also formulate the reversible and Hamiltonian cases separately.

Corollary 3.2 *Let $X_H + Y + Z$ be the sum of three reversible vector fields with Hamiltonian vector field X_H defined by* (3.2)*,*

$$Y = g_1(x,y,t)\frac{\partial}{\partial x} + g_2(x,y,t)\frac{\partial}{\partial y} \qquad \textit{and} \qquad Z = f_1(x,y,t)\frac{\partial}{\partial x} + f_2(x,y,t)\frac{\partial}{\partial y}.$$

Suppose that $a(t)$ and $b(t)$ in (3.2) *and $g_1(x,y,t)$, $g_2(x,y,t)$, $f_1(x,y,t)$, $f_2(x,y,t)$ are quasi-periodic in t with common frequency vector $\Omega = \Omega_1$ satisfying the Bruno condition* (1.4)*. Assume furthermore* $\mathrm{av}(g_1) = \mathrm{av}(g_2) = 0$ *and that g_1 has degree strictly larger than $\sigma + \alpha$ in (x,y), the coefficient function g_2 has degree strictly larger than $\sigma + \beta$ and f_1 and f_2 have degrees strictly larger than $\delta - \beta = 2\sigma + \alpha$ and $\delta - \alpha = 2\sigma + \beta$ in (x,y), respectively. If* $\mathrm{av}(a) \cdot \mathrm{av}(b) \neq 0$*, then the trivial solution $(x(t), y(t)) \equiv (0,0)$ of the system* (3.1) *is stable if and only if* $\mathrm{av}(a) \cdot \mathrm{av}(b) > 0$.

Proof. The conditions on g_1, g_2 make Y a quasi-periodic vector field of order strictly larger than σ with $\mathrm{av}(Y) = 0$ and the conditions on f_1, f_2 make Z a quasi-periodic vector field of order stricly larger than $\delta - 1$. Hence, Theorem 3.1 applies. □

Example 3.3 For $m, n \geq 1$ let the system

$$\begin{cases} \dot{x} = w_1(t)x^{2n+1} + a(t)y^{2m+1} + f_1(x,y,t) \\ \\ \dot{y} = -b(t)x^{2n+1} + w_2(t)y^{2m+1} + f_2(x,y,t) \end{cases} \tag{3.4}$$

have even quasi-periodic coefficient functions a, b and odd quasi-periodic coefficient functions w_1, w_2; furthermore $f_1(x,y,t) = f_1(-x,y,-t)$ is quasi-periodic in t and has degree strictly larger than $\delta - \beta$ in (x,y) and $f_2(x,y,t) = -f_2(-x,y,-t)$ is quasi-periodic in t and has degree strictly larger than $\delta - \alpha$. Also, $\mathrm{av}(a) \cdot \mathrm{av}(b) \neq 0$ and the common frequency vector $\Omega = \Omega_1$ of a, b, w_1, w_2, f_1 and f_2 satisfies the Bruno condition (1.4). *Then the trivial solution $(x(t), y(t)) \equiv (0,0)$ of the system* (3.4) *is stable if and only if* $\mathrm{av}(a) \cdot \mathrm{av}(b) > 0$.

Indeed, the quasi-periodic Hamiltonian vector field X_H is reversible because a and b are even functions in t. Since w_1 and w_2 are odd in t the quasi-periodic vector field Y with coefficient functions $g_1(x,y,t) = w_1(t)x^{2n+1}$ and $g_2(x,y,t) = w_2(t)y^{2m+1}$ is reversible and has average $\mathrm{av}(Y) = 0$. The order of Y is

$$\delta - 2\max(\alpha, \beta) = 2\min(n\alpha, m\beta) < \delta - 1$$

and strictly larger than

$$\sigma = n\alpha + \frac{\alpha - \beta}{2} = m\beta + \frac{\beta - \alpha}{2}$$

because $m, n \geq 1$. The conditions on f_1, f_2 make Z a quasi-periodic reversible vector field of order strictly larger than $\delta - 1$. Hence, Corollary 3.2 applies.

Remark 3.4 The smallness condition on f_1 and f_2 in Corollary 3.2 (and Example 3.3) is in particular satisfied if $f_1, f_2 = O(|x|^{n+1} + |y|^{m+1})^2$ in a neighbourhood of $(x, y) = (0, 0)$.

Corollary 3.5 *Let $H+G+F$ be the sum of the non-degenerate quasi-homogeneous Hamiltonian* (3.2)*, a quasi-homogeneous Hamiltonian G satisfying* $\mathrm{av}(G) = 0$ *with degree strictly larger than $\tau := \frac{1}{2}(\delta + 1)$ and a Hamiltonian*

$$F(x, y, t) = o(|x|^{n+1} + |y|^{m+1})^2 \quad \text{as } x^2 + y^2 \to 0.$$

Let furthermore a, b, G and F be quasi-periodic in t with common frequency vector $\Omega = \Omega_1$ satisfying the Bruno condition (1.4)*. Then the trivial solution $(x(t), y(t)) \equiv (0, 0)$ of the system with Hamiltonian $H + G + F$ is stable if and only if* $\mathrm{av}(a) \cdot \mathrm{av}(b) > 0$.

Proof. The Hamiltonian vector field $Y := X_G$ is quasi-periodic with $\mathrm{av}(Y) = 0$ and of order strictly larger than $\tau - 1 = \sigma$. The Hamiltonian F of the quasi-periodic vector field $Z := X_F$ has degree strictly larger than δ. Hence, Theorem 3.1 applies. □

Remark 3.6 Note that (1.3) and (1.2) are special cases of the systems treated in Corollaries 3.2 and 3.5, respectively. The results in [9] indicate that the formulation of Theorem 3.1 can still be improved, for instance if $-ab < \mathrm{av}(\mu) < ab$ then

$$H(x, y, t) = \frac{a}{4}y^4 + \frac{\mu(t)}{2}x^2y^2 + \frac{b}{4}x^4$$

is a definite homogeneous Hamiltonian for which

$$Y = \mu(t)x^2y\frac{\partial}{\partial x} - \mu(t)xy^2\frac{\partial}{\partial y}$$

need not have zero average.

3.2 Action angle variables

Without loss of generality we may assume that $\mathrm{av}(a) > 0$. Indeed, if $\mathrm{av}(a) < 0$ we simply reverse time which leads to the Hamiltonian $-H(x, y, -t)$ and vector fields $-Y(x, y, -t)$, $-Z(x, y, -t)$ still satisfying the conditions of Theorem 3.1.

For $\mathrm{av}(b) > 0$ all orbits of the autonomous planar system (3.3) are periodic. Denoting by $(S(t), C(t))$ the solution of (3.3) with initial conditions $S(0) = 0$ and $C(0) = 1$ and by $T_0 > 0$ the minimal period we define the transformation

$$\begin{array}{rccc} \Phi: & \mathbb{R}^+ \times \mathbb{T} & \longrightarrow & \mathbb{R}^2 \\ & (\rho, \varphi) & \mapsto & (x, y) \end{array} \tag{3.5}$$

by means of $x = c^\alpha \rho^\alpha S(\varphi T_0)$ and $y = c^\beta \rho^\beta C(\varphi T_0)$ where $c = \frac{m+n+2}{(n+1)\,\mathrm{av}(a) T_0}$. Since $(S(-t), C(-t)) = (-S(t), C(t))$ this transformation is equivariant with respect to the involutions $(\rho, \varphi) \mapsto (\rho, -\varphi)$ and $(x, y) \mapsto (-x, y)$, and because

$$\frac{\mathrm{av}(a)}{2m+2} C^{2m+2}(t) + \frac{\mathrm{av}(b)}{2n+2} S^{2n+2}(t) = \mathrm{av}(H)(0,1) = \frac{\mathrm{av}(a)}{2m+2}$$

the mapping Φ is area-preserving (i.e. canonical or symplectic). In fact,

$$\begin{aligned}
\left|\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right| &= \left|\alpha T_0 \,\mathrm{av}(b) c^{\alpha+\beta} \rho^{\alpha+\beta-1} S^{2n+2}(\varphi T_0) + \beta T_0 \,\mathrm{av}(a) c^{\alpha+\beta} \rho^{\alpha+\beta-1} C^{2m+2}(\varphi T_0)\right| \\
&= \frac{1}{m+n+2} \cdot \left| T_0 c \cdot \Big((m+1)\,\mathrm{av}(b) S^{2n+2}(\varphi T_0) + (n+1)\,\mathrm{av}(a) C^{2m+2}(\varphi T_0) \Big) \right| \\
&= 1.
\end{aligned}$$

Hence, Φ transforms conservative vector fields into conservative vector fields. In particular, the system (3.3) is transformed into

$$\begin{aligned}
\dot\rho &= -\frac{\partial h_0}{\partial \varphi} = 0 \\
\dot\varphi &= \frac{\partial h_0}{\partial \rho} = d \cdot \rho^{\delta - 1}
\end{aligned}$$

with Hamiltonian

$$h_0(\rho, \varphi, t) = h_0(\rho) = \frac{c^\delta \,\mathrm{av}(a)}{2m+2} \rho^\delta = \frac{d \cdot \rho^\delta}{\delta}$$

where we write $d = \beta c^\delta \mathrm{av}(a)$. The original vector field $X_H + Y + Z$ is transformed into

$$X_{\Phi^* H} + \Phi^* Y + \Phi^* Z \tag{3.6}$$

with

$$H(\Phi(\rho,\varphi), t) = h_0(\rho) + h_1(\rho, \varphi, t).$$

Here

$$h_1(\rho, \varphi, t) = c^\delta \rho^\delta \left[\frac{a_1(t)}{2m+2} C^{2m+2}(\varphi T_0) + \frac{b_1(t)}{2n+2} S^{2n+2}(\varphi T_0) \right] \tag{3.7}$$

where we have split

$$a(t) = \mathrm{av}(a) + a_1(t), \quad b(t) = \mathrm{av}(b) + b_1(t)$$

into the constant averages and the time-dependent parts with averages

$$\mathrm{av}(a_1) = 0, \quad \mathrm{av}(b_1) = 0.$$

Furthermore

$$\Phi^* Y(\rho, \varphi, t) = \xi_1(\rho, \varphi, t) \frac{\partial}{\partial \rho} + \xi_2(\rho, \varphi, t) \frac{\partial}{\partial \varphi}$$

where the coefficient functions ξ_i are both quasi-periodic in t with zero average and

$$\xi_1 = o(\rho^\tau), \quad \xi_2 = o(\rho^\sigma) \qquad \text{as } \rho \to 0$$

since Y is of order strictly larger than $\tau - 1 = \sigma$. In other words, Y is of order $\sigma + \nu$ and we have

$$\xi_1 = O(\rho^{\tau+\nu}), \quad \xi_2 = O(\rho^{\sigma+\nu})$$

for sufficiently small $\nu > 0$. Since Z is of order strictly larger than $\delta - 1 = 2\sigma$ we similarly obtain

$$\Phi^* Z(\rho, \varphi, t) = O(\rho^{\delta+\nu})\frac{\partial}{\partial \rho} + O(\rho^{2\sigma+\nu})\frac{\partial}{\partial \varphi},$$

shrinking ν a bit if necessary to ensure that Z is of order $2\sigma + \nu$.

Remark 3.7 For $m = n = 0$ the transformation Φ is defined in terms of $S(t) = \sin \kappa t$ and $C(t) = \cos \kappa t$ where $\kappa = \sqrt{\mathrm{av}(a) \cdot \mathrm{av}(b)}$ is the frequency of the harmonic oscillator (3.3).

3.3 Dynamical stability

The transformed vector field (3.6) has equations of motion

$$\begin{array}{rcl} \dot{\rho} & = & \eta_1(\rho, \varphi, t) + \zeta_1(\rho, \varphi, t) \\ \dot{\varphi} & = & d \cdot \rho^{\delta-1} + \eta_2(\rho, \varphi, t) + \zeta_2(\rho, \varphi, t) \end{array} \tag{3.8}$$

with expressions at the right hand side defined by

$$\begin{array}{rcl} X_{h_1} + \Phi^* Y & =: & \eta_1 \frac{\partial}{\partial \rho} + \eta_2 \frac{\partial}{\partial \varphi} \\ \Phi^* Z & =: & \zeta_1 \frac{\partial}{\partial \rho} + \zeta_2 \frac{\partial}{\partial \varphi}. \end{array}$$

The coefficient functions η_1 and η_2 have zero average in t. This implies that the indefinite integrals

$$\int \eta_1(\rho, \varphi, t) dt \quad \text{and} \quad \int \eta_2(\rho, \varphi, t) dt$$

are again quasi-periodic and can be used to define a time-dependent transformation Ψ_t. To ensure that the transformed system is again conservative we define Ψ_t as the time-1-mapping of the 'auxiliary' vector field $W = -X_{h_1} - \Phi^* Y$. The transformation

$$(\varrho, \vartheta) = \Psi_t(\rho, \varphi) =: (\rho, \varphi) + \psi(\rho, \varphi, t)$$

with

$$\frac{\partial \psi(\rho, \varphi, t)}{\partial t} = -\eta(\varrho, \vartheta, t)$$

turns (3.8) into

$$\begin{array}{rcl} \dot{\varrho} & = & \eta_1 + \zeta_1 + \frac{\partial \psi_1}{\partial \rho}\dot{\rho} + \frac{\partial \psi_1}{\partial \varphi}\dot{\varphi} + \frac{\partial \psi_1}{\partial t} = O(\varrho^{\delta+\nu}) \\ \dot{\vartheta} & = & d \cdot \varrho^{\delta-1} + \eta_2 + \zeta_2 + \frac{\partial \psi_2}{\partial \rho}\dot{\rho} + \frac{\partial \psi_2}{\partial \varphi}\dot{\varphi} + \frac{\partial \psi_2}{\partial t} = d \cdot \varrho^{2\sigma} + O(\varrho^{2\sigma+\nu}) \end{array} \tag{3.9}$$

and we can apply the results from Section 2. To this end we expand around a fixed small value $\varrho = r\varepsilon$ with $r \in [\frac{1}{2}, \frac{3}{2}]$ and scale the translated variable by ε^κ with

$$\kappa = \frac{\nu}{2\delta - 1},$$

cf. [2]. This amounts for every $0 < \varepsilon < \varepsilon^*$ to the transformation

$$\begin{array}{rccc} \Theta_\varepsilon : & (\frac{-r}{\varepsilon^\kappa}, \frac{r}{\varepsilon^\kappa}) \times \mathbb{T} & \longrightarrow & \mathbb{R}^+ \times \mathbb{T} \\ & (I, \theta) & \mapsto & (\varrho, \vartheta) \end{array}$$

defined by $\varrho = \varepsilon(r + \varepsilon^\kappa I)$, $\vartheta = \theta$ and turns (3.9) into

$$\begin{aligned} \dot{I} &= O(\varepsilon^{\delta+\nu-1-\kappa}) = O(\varepsilon^{2\sigma(1+2\kappa)}) \\ \dot{\theta} &= dr^{2\sigma}\varepsilon^{2\sigma} + O(\varepsilon^{2\sigma(1+\kappa)}). \end{aligned}$$

Rescaling time by $\varepsilon^{2\sigma}$ then yields

$$\begin{aligned} \dot{I} &= O(\varepsilon^{4\sigma\kappa}) \\ \dot{\theta} &= A + O(\varepsilon^{2\sigma\kappa}) \end{aligned}$$

in the form (2.1) or (2.26) with parameter

$$A = dr^{2\sigma} = \frac{1}{T_0^\delta}\left(\frac{r}{\beta \operatorname{av}(a)}\right)^{2\sigma} \in \mathbb{R}, \quad r \in [\frac{1}{2}, \frac{3}{2}].$$

Here we can choose $s = \varepsilon^{\sigma\kappa}$ to apply Theorems 2.1 and 2.3, then $s^2 = \varepsilon^{2\sigma\kappa}$ and the right hand sides fulfill (2.2) or (2.27), respectively. The resulting quasi-periodic solutions determine invariant cylinders around the t-axis, confining integral curves and thus implying stability of the orbit along the t-axis.

Remark 3.8 For $m = n = 0$ this problem of small twist can be solved with the help of the higher order term in the right hand side of $\dot{\varphi}$, using a mild non-degeneracy condition on ζ_2 (and thus on Z).

3.4 Dynamical instability

Still assuming $\operatorname{av}(a) > 0$ we now consider the case $\operatorname{av}(b) < 0$. To avoid negative constants we define $\varpi := -\operatorname{av}(b) > 0$ whence

$$a(t) = \operatorname{av}(a) + a_1(t) \quad \text{and} \quad b(t) = -\varpi + b_1(t)$$

with $\operatorname{av}(a_1) = \operatorname{av}(b_1) = 0$. Replacing (3.3) by an 'auxiliary' system

$$\left\{ \begin{aligned} \dot{x} &= \operatorname{av}(a) y^{2m+1} \\ \dot{y} &= -\varpi x^{2n+1} \end{aligned} \right. \tag{3.10}$$

we obtain again a periodic flow defining the transformation Φ in (3.5). The Hamiltonian (3.2) now turns into

$$\Phi^* H(\rho, \varphi, t) = \frac{\rho^\delta}{\delta} d(\varphi) + h_1(\rho, \varphi, t).$$

with h_1 still given by (3.7) and

$$d(\varphi) \quad = \quad \beta c^\delta \operatorname{av}(a) C^{2m+2}(\varphi T_0) - \alpha c^\delta \varpi S^{2n+2}(\varphi T_0)$$

where in § 3.2 the expression $d(\varphi) \equiv \beta c^\delta \operatorname{av}(a)$ was φ-independent. The real analytic function d is 1-periodic in φ, and from

$$\beta c^\delta \operatorname{av}(a) C^{2m+2}(t) + \alpha c^\delta \varpi S^{2n+2}(t) \equiv \beta c^\delta \operatorname{av}(a) \tag{3.11}$$

we deduce

$$d(\varphi) = 2\alpha c^\delta \varpi \left[\frac{\beta \operatorname{av}(a)}{2\alpha\varpi} - S^{2n+2}(\varphi T_0) \right].$$

The original vector field $X_H + Y + Z$ is transformed into $X_{\Phi^* H} + \Phi^* Y + \Phi^* Z$ with equations of motion

$$\begin{aligned} \dot{\rho} &= \tfrac{\rho^\delta}{\delta} d'(\varphi) + \eta_1(\rho, \varphi, t) + \zeta_1(\rho, \varphi, t) \\ \dot{\varphi} &= \rho^{\delta-1} d(\varphi) + \eta_2(\rho, \varphi, t) + \zeta_2(\rho, \varphi, t). \end{aligned} \tag{3.12}$$

The expressions at the right hand side are again defined by

$$\begin{aligned} X_{h_1} + \Phi^* Y &=: \eta_1 \frac{\partial}{\partial \rho} + \eta_2 \frac{\partial}{\partial \varphi} \\ \Phi^* Z &=: \zeta_1 \frac{\partial}{\partial \rho} + \zeta_2 \frac{\partial}{\partial \varphi} \end{aligned}$$

and still satisfy

$$\eta_1 = o(\rho^\tau), \quad \eta_2 = o(\rho^\sigma), \quad \zeta_1 = o(\rho^\delta), \quad \zeta_2 = o(\rho^{2\sigma}) \qquad \text{as } \rho \to 0.$$

As in § 3.3 we use the time-t-mapping Ψ_t of the 'auxiliary' vector field

$$W(\rho, \varphi, t) = -\eta_1(\rho, \varphi, t) \frac{\partial}{\partial \rho} - \eta_2(\rho, \varphi, t) \frac{\partial}{\partial \varphi}$$

to transform (3.12) into

$$\begin{aligned} \dot{\varrho} &= -\tfrac{\varrho^\delta}{\delta} d'(\vartheta) + o(\varrho^\delta) \\ \dot{\vartheta} &= \varrho^{\delta-1} d(\vartheta) + o(\varrho^{\delta-1}) \end{aligned} \tag{3.13}$$

preserving the conservative nature (reversible and/or Hamiltonian) of the system.

The phase portrait of (3.10) is reversible with respect to the reflections about the x-axis and about the y-axis, whence both S and C are positive on $(0, \frac{1}{4}T_0)$ with $S(0) = 0 = C(\frac{1}{4}T_0)$ at the boundary. In particular

$$d(0) = \beta c^\delta \operatorname{av}(a), \quad d(\frac{1}{4}) = -\beta c^\delta \operatorname{av}(a)$$

because of (3.11) and

$$d'(\vartheta) = -2\delta c^\delta \operatorname{av}(a) \varpi \, T_0 \, S^{2n+1}(\vartheta T_0) \, C^{2m+1}(\vartheta T_0)$$

is negative for all $\vartheta \in (0, \frac{1}{4})$. Therefore d has a unique root $\vartheta^* \in (0, \frac{1}{4})$ and there is an open interval $(\vartheta^* - \varsigma, \vartheta^* + \varsigma) \subseteq (0, \frac{1}{4})$ on which d is monotonously decreasing with postive values on $(\vartheta^* - \varsigma, \vartheta^*)$ and negative values on $(\vartheta^*, \vartheta^* + \varsigma)$. On the region

$$\Lambda = \{(\varrho, \vartheta, t) \in \mathbb{R}^3 : \varrho > 0, \vartheta^* - \varsigma < \vartheta < \vartheta^* + \varsigma\}$$

we consider the function

$$V = \varrho^{\delta} \sin\phi \quad \text{with} \quad \phi = \frac{\pi}{2\varsigma}(\vartheta - \vartheta^* + \varsigma).$$

Then $V > 0$ in Λ and $V = 0$ on $\partial\Lambda$, the boundary of Λ. It follows that

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(3.13)} &= \frac{\partial V}{\partial \varrho}\dot{\varrho} + \frac{\partial V}{\partial \vartheta}\dot{\vartheta} + \frac{\partial V}{\partial t} \\ &= \varrho^{2\delta-1}\left[-d'(\vartheta)\sin\phi + \frac{\pi}{2\varsigma}d(\vartheta)\cos\phi\right] + o(\varrho^{2\delta-1}). \end{aligned}$$

For $\vartheta \in (\vartheta^*, \vartheta^* + \varsigma)$, we have $\frac{\pi}{2} < \frac{\pi}{2\varsigma}(\vartheta - \vartheta^* + \varsigma) = \phi < \pi$ so that $\cos\phi < 0$. It follows that $-d'(\vartheta)\sin\phi + \frac{\pi}{2\varsigma}d(\vartheta)\cos\phi > 0$ for $\vartheta \in (\vartheta^*, \vartheta^* + \varsigma)$. Similarly, we also have $-d'(\vartheta)\sin\phi + \frac{\pi}{2\varsigma}d(\vartheta)\cos\phi > 0$ for $\vartheta \in (\vartheta^* - \varsigma, \vartheta^*)$. Since $-d'(\vartheta^*) > 0$ we conclude that $\left.\frac{dV}{dt}\right|_{(3.13)} > 0$ on Λ, if ϱ is sufficiently small. It follows from Chetaev's theorem [6] that the trivial solution of (3.1) is unstable. □

Remark 3.9 For $m = n = 0$ the hyperbolic linear system (3.3) implies dynamical instability of $(x, y) = (0, 0)$ for $X_H + Z$ as well.

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References

[1] V.I. Arnol'd, S.M. Gusein-Zade and A.N. Varchenko, *Singularities of Differentiable Maps, Vol. 1*, Birkhäuser (1985)

[2] Yu.N. Bibikov, Stability of Zero Solutions of Essentially Nonlinear One-Degree-Of-Freedom Hamiltonian and Reversible Systems, *Differential Equations* **38** (2002) 609–614

[3] A. Berretti and G. Gentile, Bryuno Function and the Standard Map, *Comm. Math. Phys.* **220** (2001) 623–656

[4] H.W. Broer, G.B. Huitema and M.B. Sevryuk, *Quasi-Periodic Motions in Families of Dynamical Systems: Order amidst Chaos*, LNM **1645**, Springer (1996)

[5] T. Carletti and S. Marmi, Linearization of analytic and non-analytic germs of diffeomorphisms of $(\mathbb{C}, 0)$, *Bull. Soc. Math. France* **128** (2000) 69–85

[6] N. Chetaev, *The stability of motion*, Oxford: Pergamon (1961)

[7] B. Liu, The stability of the equilibrium of a conservative system, *J. Math. Anal. Appl.* **202** (1996) 133–149

[8] B. Liu, The stability of the equilibrium of reversible systems, *Trans. Amer. Math. Soc.* **351** (1999) 515–531

[9] B. Liu, The Stability of the Equilibrium of Planar Hamiltonian and Reversible systems, *J. Dynamics Diff. Eq.* **18** (2006) 975–990

[10] J. Moser, Convergent Series Expansion for Quasi-Periodic Motions, *Math. Ann.* **169** (1967) 136–176

[11] J. Moser, *Stable and Random Motions in Dynamical Systems: With Special Emphasis on Celestial Mechanics*, Princeton Univ. Press (1973)

[12] D. Núñez and R. Ortega, Parabolic fixed point and stability criteria for nonlinear Hill's equation, *Z. angew. Math. Phys.* **51** (2000) 890–911

[13] R. Ortega, The stability of the equilibrium of a nonlinear Hill's equation, *SIAM J. Math. Anal.* **25** (1994) 1394–1401

[14] R. Pérez-Marco and J.-C. Yoccoz, Germes de feulletages holomorphes à holomie prescrite, *Astérisque* **222** (1994) 345–371

[15] J. Pöschel, A lecture on the classical KAM theorem, *Proc. Symp. Pure Math.* **69** (2001) 707–732

[16] H. Rüssmann, Invariant tori in non-degenerate nearly integrable Hamiltonian systems, *Reg. & Chaot. Dyn.* **6** (2001) 119–204

[17] H. Rüssmann, Stability of elliptic fixed points of analytic area-preserving mappings under the Bruno condition, *Ergod. Th. & Dynam. Sys.* **22** (2002) 1551–1573

[18] C.L. Siegel and J.K. Moser, *Lectures on Celestial Mechanics*, Springer (1971)

[19] J.-C. Yoccoz, Linéarization des germes de difféomorphismes holomorphes de $(\mathbb{C}, 0)$, *C. R. Acad. Sci. Paris* **306** (1988) 55–58

[20] J.-C. Yoccoz, Petit diviseurs en dimension 1, *Astérisque* **231** (1995)