Non-degeneracy conditions in KAM theory

Dedicated to the memory of Floris Takens

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Abstract

Persistence of invariant tori in a perturbed dynamical system requires two kinds of conditions to be met. A strong non-resonance condition ensures a dense quasi-periodic orbit on both the unperturbed and the perturbed torus. A non-degeneracy condition enforces a sufficiently large subset of the unperturbed tori to be non-resonant and thus yields persistence. In the past 60 years various such conditions have been formulated and a number of them are reviewed here.

1 Introduction

First results on persistence of invariant tori carrying quasi-periodic dynamics were achieved for Hamiltonian systems. Kolmogorov [31] studied how integrable systems

$$\begin{aligned} \dot{x} &= \omega(y) \\ \dot{y} &= 0 \end{aligned} \tag{1}$$

behaved under small (Hamiltonian) perturbations. In (1) the integrable system is already expressed in action angle variables $(x, y) \in \mathbb{T}^n \times \mathbb{R}^n$ (with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) whence (1) can be solved straightforwardly. The solution curves $(x(t), y(t)) = (x + t\omega(y), y)$ are conditionally periodic with frequency vector $\omega(y)$, spinning around invariant tori. Non-resonant tori $\mathbb{T}^n \times \{y\}$ have a frequency vector $\omega = \omega(y)$ with rationally independent components, thus satisfying

$$\bigwedge_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \langle k \mid \omega \rangle \;:=\; k_1\omega_1 + \ldots + k_n\omega_n \;\neq\; 0 \;.$$

This ensures that $\mathbb{T}^n \times \{y\}$ consists of dense quasi-periodic orbits $(x + t\omega(y), y)$ and therefore is a *dynamical object*: closed, invariant and minimal with these properties. A resonance $\langle k \mid \omega \rangle = 0$, $k \neq 0$ makes $\mathbb{T}^n \times \{y\}$ a disjoint union of invariant $(n-1)$–tori and thus unlikely to persist as an invariant n–torus.

The simplest Hamiltonian system with conditionally periodic motion is the 2–dimensional harmonic oscillator

$$\begin{array}{rcl} \dot{q}_i & = & \omega_i p_i \\ \dot{p}_i & = & -\omega_i q_i \end{array} \quad , \quad i = 1, 2 \tag{2}$$

which can be brought into the form (1) by means of symplectic polar co-ordinates $q_i = \sqrt{2y_i} \cos x_i$ and $p_i = \sqrt{2y_i} \sin x_i$ (excluding $\{y_i = \frac{1}{2}(p_i^2 + q_i^2) = 0\}$, in particular excluding the equilibrium at the origin). Here $\omega(y) \equiv (\omega_1, \omega_2)$ does not depend on y, so either all motions are periodic or all motions are quasi-periodic. This makes the invariant 2–tori also in the quasi-periodic case vulnerable to perturbations. Indeed, a perturbation may perform the two-step programme of first turning the frequency ratio $\frac{\omega_1}{\omega_2}$ rational and then destroying the resulting resonant tori.

In frequency space, both the resonant and the non-resonant frequency vectors form dense subsets. The condition

$$\det D\omega \;\; \neq \;\; 0 \tag{3}$$

of Kolmogorov [31] similarly makes the phase space of (1) a disjoint union of two dense sets. Indeed, the mapping $\omega : \mathbb{Y} \longrightarrow \mathbb{R}^n$ assigns the frequency vector $\omega(y)$ to each invariant torus $\mathbb{T}^n \times \{y\}$, y varying in the domain $\mathbb{Y} \subseteq \mathbb{R}^n$, and Kolmogorov's non-degeneracy condition (3) makes this *frequency mapping* a local diffeomorphism. In this way the whole geometry of the frequency space $\mathbb{R}^n$ is pulled back into the phase space $\mathbb{T}^n \times \mathbb{Y}$.

To ensure persistence, an open neighbourhood of the resonant frequency vectors has to be avoided. At the same time the remaining — topologically small — set of far-from-resonance frequency vectors should be as large as possible, making persistence of strongly non-resonant invariant tori meaningful. One way to simultaneously achieve these two goals is to impose *Diophantine conditions*

$$\bigwedge_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \quad |\langle k \mid \omega \rangle| \;\; \geq \;\; \frac{\gamma}{|k|^\tau} \tag{4}$$

on the frequency vectors, where $|k| := |k_1| + \ldots + |k_n|$. Indeed, for fixed $\tau > n - 1$ the relative measure of the set of excluded frequency vectors (not satisfying (4)) is of order $\mathcal{O}(\gamma)$ as $\gamma \to 0$ — the topologically small set $\mathbb{R}^n_{\gamma,\tau}$ of Diophantine frequency vectors is measure-theoretically large. A typical choice is $\gamma = \sqrt{\varepsilon}$ where ε measures the size of the perturbation. For $n = 2$ the *Cantor set* of Diophantine frequency vectors is sketched in figure 1. These notions allow to formulate the follwing result, cf. [31, 1, 36, 18, 39]. Here and further on the smallness conditions are a bit lacking in precision[1] for the sake of a simpler presentation.

Theorem 1 (Kolmogorov, Arnol'd, Moser, Chierchia, Gallavotti, Pöschel) *Let $H = N + P$ be a real analytic Hamiltonian on $\mathbb{T}^n \times \overline{\mathbb{Y}}$, $\mathbb{Y} \subseteq \mathbb{R}^n$ a relative compact domain, with integrable $N = N(y)$ satisfying the Kolmogorov condition* $\det D^2 N \neq 0$ *and uniformly*

[1]To be more precise, one has to extend the perturbed Hamiltonian H to a complex neighbourhood of $\mathbb{T}^n \times \overline{\mathbb{Y}}$ and the uniform bound $\|P\|_\infty < \varepsilon$ has to hold on that neighbourhood.

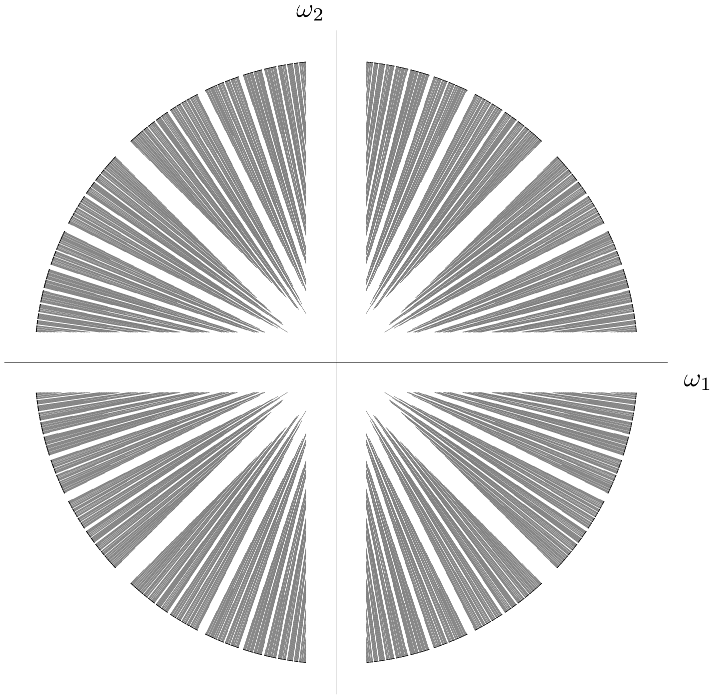

Figure 1: The Cantor set of Diophantine frequency vectors in the plane.

bounded perturbation $P = P(x, y)$. Then there exists $\varepsilon > 0$ such that for all $\|P\|_\infty < \varepsilon$ there is a canonical transformation ϕ near the identity and a measure-theoretically large Cantor set $\mathbb{Y}' \subseteq \mathbb{Y}$ with the property that for $y \in \mathbb{Y}'$ the transformed Hamiltonian $H \circ \phi$ does not depend on $x \in \mathbb{T}^n$.

While the Cantor set $\mathbb{T}^n \times \mathbb{Y}'$ consists of invariant tori with quasi-periodic flow, no statement is made for $y \in \mathbb{Y} \backslash \mathbb{Y}'$.

Sketch (caricature) of proof. A canonical transformation φ solving the *homological equation*

$$H \circ \varphi \;=\; N + \int_{\mathbb{T}^n} P \, \mathrm{d}x \tag{5}$$

would reveal the perturbed Hamiltonian H to be integrable, but also prove the theorem. Instead of solving (5) directly we work with the linearized version

$$N(y) + P(x, y) - \langle \omega(y) \mid \partial_x W(x, y) \rangle \;=\; N(y) + P_0(y) \tag{6}$$

of this equation. Here the Hamiltonian function W is searched for — the transformation φ will be the time–1–mapping of the Hamiltonian system

$$\begin{aligned} \dot{x} &= \partial_y W \\ \dot{y} &= -\partial_x W \end{aligned}$$

defined by W — while $\omega = DN$ is the frequency mapping and P_0 is the $\mathbb{T}^n$–average of P, *i.e.* the 0–coefficient in the Fourier series

$$P(x, y) \quad = \quad \sum_{k \in \mathbb{Z}^n} P_k(y)\, \mathrm{e}^{2\pi\mathrm{i}\langle k|x\rangle} \quad .$$

Developing the unknown W in a Fourier series as well turns the linearized homological equation (6) into

$$\bigwedge_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \quad 2\pi\mathrm{i}\langle k \mid \omega(y)\rangle W_k(y) \quad = \quad P_k(y) \quad , \tag{7}$$

the formal solution of which copes with the notorious small denominators. The Diophantine conditions (4) provide a polynomial bound $\sim |k|^\tau$ of these, so that (for $y \in \omega^{-1}(\mathbb{R}^n_{\gamma,\tau})$ fixed) the exponentially decaying coefficients of the real analytic P make the W_k decay exponentially as well and result in a function W that is real analytic in x.

The errors made by solving (6) instead of (5) are collected in a new perturbation R and one of the things an actual proof has to show is that R is indeed smaller than the original perturbation P — so much smaller that the iterative scheme ensuing from solving (6) with P replaced by R converges and yields the desired ϕ. The resonance gaps left open in the definition of ϕ are finally filled using Whitney's Extension Theorem — here ϕ does not have to achieve anything to prove the theorem. $\square$

In fact, for this Newton-like scheme one expects quadratic convergence, *i.e.* $\|R\|_\infty \sim \|P\|^2_\infty$; see [44, 41] for a slowly converging iteration scheme. The necessary estimates involve the Cauchy formula whence the original formulation was for real analytic Hamiltonians. Using an interspersed approximation by holomorphic functions at each iteration step Moser [36] extended the validity to Hamiltonians that are only finitely often differentiable; the initial regularity of 333 derivatives was subsequently brought down. Takens [46] provided a lower bound by means of a C^1–counterexample.

Broer and Takens [17] showed that the persisting tori are essentially unique. This allows to patch together the local[2] conjugacies of theorem 1 between the flows defined by N and H to a global conjugacy, see [7].

The next section is concerned with tori whose normal behaviour is trivial, as in (1). Section 3 then addresses tori where non-trivial normal dynamics may interact with the quasi-periodic motion and additional possible resonances have to be taken care of. In the final section a formulation of non-degeneracy (called versality) is given that also applies to invariant tori undergoing a bifurcation.

2 Maximal tori

The integrability of the Hamiltonian system (1) stems from the x–independence of the corresponding Hamiltonian function $N(x, y) \equiv N(y)$. This motivates the following extension

[2]The necessary action angle variables for the integrable Hamiltonian N can always be constructed by shrinking $\mathbb{Y}$ where necessary.

of this notion.

Definition 2 (Broer, Huitema, Takens) *A vector field on $\mathbb{T}^n \times \mathbb{Y} \subseteq \mathbb{T}^n \times \mathbb{R}^m$ is called integrable if it is equivariant with respect to the $\mathbb{T}^n$–action $(x, y) \mapsto (x + \xi, y)$.*

In the simplest situation $m = 0$ it is not persistence of the invariant torus that is in question — this is the whole phase space — but of the quasi-periodic dynamics defined by $\dot{x} = \omega$. Again a single frequency vector is vulnerable to perturbations and one has to consider parameter-dependent families of vector fields on $\mathbb{T}^n$. The most transparent situation is where $\omega \in \mathbb{R}^n$ itself plays the rôle of parameter.

Theorem 3 (Arnol'd, Moser, Herman, Broer, Huitema, Takens) *Let the perturbed vector field $(\omega + f(x; \omega))\partial_x$ be real analytic on $\mathbb{T}^n$, with ω varying in the closure $\overline{\mathbb{O}}$ of a relative compact domain $\mathbb{O} \subseteq \mathbb{R}^n$. Then there exists $\varepsilon > 0$ such that for all $\|f\|_\infty < \varepsilon$ there is a diffeomorphism*

$$
\begin{array}{rccc}
\Phi \; : & \mathbb{T}^n \times \mathbb{O} & \longrightarrow & \mathbb{T}^n \times \mathbb{O} \\
 & (x, \omega) & \mapsto & (\phi(x, \omega), \varphi(\omega))
\end{array}
$$

near the identity and a measure-theoretically large Cantor set $\mathbb{O}' \subseteq \mathbb{O}$ with the property that the restriction of Φ to $\mathbb{T}^n \times \mathbb{O}'$ conjugates $\omega\partial_x$ with $(\omega + f)\partial_x$.

Direct proofs are given in [15, 41]. Kolmogorov's non-degeneracy condition (3) can be used here as well, to allow for more general frequency mappings $\omega : \mathbb{O} \longrightarrow \mathbb{R}^n$ than just the identity. In the dissipative case of theorem 3 there is no intrinsic reason for the parameter domain $\mathbb{O}$ to have the same dimension as the torus $\mathbb{T}^n$, so one can easily generalize to domains $\mathbb{O} \subseteq \mathbb{R}^s$ with $s \geq n$ and require the frequency mapping to be a submersion. Below we discuss generalizations that lower (instead of increase) the number of parameters.

Reversible systems are situated in between dissipative systems (as above, with no structure to be preserved) and Hamiltonian systems. On $\mathbb{T}^n \times \mathbb{Y} \subseteq \mathbb{T}^n \times \mathbb{R}^m$, with involutive symmetry $G(x, y) = (-x, y)$, a vector field $f\partial_x + g\partial_y$ is called reversible if orbits are mapped by G to orbits with time reversed. This implies that f is even in x and g is odd in x. In particular $g(0, y) \equiv 0$ and an integrable reversible vector field is automatically in the integrated form (1). Here it is again the phase space variable y that serves as a parameter for the occurring frequencies, but now $m = n$ is only the most important of many possible cases. The reversible analogue of theorems 1 and 3 can be shortened to the following formulation.

Theorem 4 (Moser, Pöschel, Sevryuk, Huitema) *If the frequency mapping $\omega : \mathbb{Y} \longrightarrow \mathbb{R}^n$ is a submersion, then most invariant tori of* (1) *survive a small reversible perturbation.*

For proofs see [38, 39, 45, 29]. If the frequency mapping is a submersion, then the whole geometry of the set of frequency vectors satisfying the Diophantine conditions (4) is pulled back into (the factor $\mathbb{Y}$ of) phase space — or into parameter space $\mathbb{O}$ in the dissipative case. Note that if $m < n$ in the reversible case, then one can consider a parameter dependent reversible vector field whence a submersive frequency mapping pulls back into the product $\mathbb{Y} \times \mathbb{O}$ of (a factor of) phase space and parameter space. Here the phase space variables $y \in \mathbb{Y}$ act as parameters that are distinguished with respect to the external parameters from $\mathbb{O}$.

The Cantor set resulting from pulling back (4) always has a continuous direction: for $\omega \in \mathbb{R}^n_{\gamma,\tau}$ and $t \geq 1$ also $t\omega \in \mathbb{R}^n_{\gamma,\tau}$. This can be used to lower the necessary dimension of the domain of the frequency mapping to $n-1$. While ω itself is no longer controlled, the frequency ratios

$$[\omega_1 : \omega_2 : \ldots : \omega_n] \quad \in \quad \mathbb{RP}^n$$

of the set of Diophantine frequency vectors form a (now totally discontinuous) Cantor set of large relative measure. Arnol'd [1] used this to formulate the condition

$$\det \begin{pmatrix} D\omega & \omega \\ \omega^T & 0 \end{pmatrix} \neq 0 \tag{8}$$

of iso-energetic non-degeneracy on the Hamiltonian system (1). If (8) holds then on every energy shell $H^{-1}(h)$ of the slightly perturbed Hamiltonian $H = N + P$ most tori survive — not with the same frequency, but with the same frequency ratio. In [13] this result was obtained from theorem 1 by means of a time-scaling. This approach allowed to prove similarly 'relaxed' corollaries of theorems 3 and 4 as well, see [16, 15].

The geometry of the set of Diophantine frequency vectors is also the key for a further decrease of the number of necessary parameters. The inequalities in (4) all exclude small open neighbourhoods of linear hyperplanes. If the frequency mapping is not a submersion, then one has to cope with a submanifold $M \subseteq \mathbb{R}^n$, describing the inherent dependencies of the components of the possible frequency vectors. What has to be avoided is that large portions of M are parallel to one of the linear hyperplanes and vanish in a resonance gap. The iso-energetic non-degeneracy condition (8) achieves this by requiring M to be transverse to the straight lines through the origin.

Next to this linear approach the resonance gaps can also be avoided by means of non-linear considerations. If the submanifold M is sufficiently bent, then if M is at some point tangent to one of the resonance hyperplanes it will soon exit the (small!) resonance gap. This similarly works for torsion and higher order derivatives. Rüssmann [42, 43] formulated this in terms of the span

$$< \frac{\partial^{|\ell|}\omega}{\partial y^\ell} \;\Bigg|\; |\ell| \leq L > \quad = \quad \mathbb{R}^n \tag{9}$$

of the partial derivatives up to order $L \in \mathbb{N}$ of the frequency mapping yielding the whole frequency space $\mathbb{R}^n$. Note that $L = 1$ corresponds to the iso-energetic non-degeneracy condition (8), while submersivity of the frequency mapping — and in particular Kolmogorov's condition (3) — amounts to restricting to $|\ell| = 1$ in (9). To ensure that the relative measure of the pull-back of the excluded frequency vectors (not satisfying (4)) is still sufficiently small one has to thicken the set of Diophantine frequency vectors a bit and require $\tau > nL - 1$ in (4). The relative measure is then of order $\mathcal{O}(\gamma^{1/L})$ as $\gamma \longrightarrow 0$, see [15, 43].

The persistence result obtained under Rüssmann's non-degeneracy condition (9), with $L \geq 2$, is weakenend as there is no longer a Cantor family of tori with prescribed frequency vectors (or ratios) that survive the perturbation. What is present in the perturbed system is a Cantor family of invariant tori with frequency vectors slightly shifted from their unperturbed counterparts. Note that all non-degeneracy conditions amount to using the frequency vectors to parametrise the invariant tori and then either require the frequency mapping to be a submersion or prevent large parts of its image to fall into resonance gaps.

The set of strongly non-resonant frequency vectors can be further thickened by replacing the denominator in (4) with $\Delta(|k|)$ using suitable Rüssmann approximation functions, homeomorphisms $\Delta : [1, \infty[\longrightarrow [1, \infty[$ satisfying

$$\int_1^\infty \frac{\ln \Delta(t)}{t^2} \, \mathrm{d}t \quad < \quad \infty \ .$$

For simplicity we keep using Diophantine conditions in the sequel.

3 Lower-dimensional tori

In [37, 16, 14, 15] the results formulated in theorems 1, 3 and 4 are unified into a single theorem on persistence of maximal tori in perturbed structure preserving vector fields. Furthermore the results are generalized to lower-dimensional tori, *i.e.* families of tori with non-trivial normal behaviour. Starting point is an integrable system

$$\begin{array}{rclcl} \dot{x} & = & f_\lambda(y,z) & = & \omega_\lambda(y) \ + \ \mathcal{O}(z) \\ \dot{y} & = & g_\lambda(y,z) & = & \mathcal{O}(z^2) \\ \dot{z} & = & h_\lambda(y,z) & = & \Omega_\lambda(y) \cdot z \ + \ \mathcal{O}(z^2) \end{array} \tag{10}$$

with $(x, y, z, \lambda) \in \mathbb{T}^n \times \mathbb{Y} \times \mathbb{V} \times \mathbb{O} \subseteq \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^s$. In the Hamiltonian case one has $m = n$, $q = 2p$ and $\Omega_\lambda(y) \in \mathfrak{sp}(2p, \mathbb{R})$, in the dissipative case $m = 0$ and $\Omega_\lambda(y) \in \mathfrak{gl}(q, \mathbb{R})$ and in the reversible case $q = 2p$ and $\Omega_\lambda(y) \in \mathfrak{gl}_-(2p, \mathbb{R})$. The latter is defined in terms of the reversor

$$G(x,y,z) \quad = \quad (-x, y, Rz) \tag{11}$$

as the set of $(2p \times 2p)$–matrices Ω with $\Omega R = -R\Omega$ and not a Lie subalgebra of $\mathfrak{gl}(2p, \mathbb{R})$, but satisfies $[\mathfrak{gl}_+, \mathfrak{gl}_-] \subseteq \mathfrak{gl}_-$ where $\mathfrak{gl}_+(2p, \mathbb{R})$ is the Lie algebra of $(2p \times 2p)$–matrices that commute with R.

To include this peculiarity of the reversible case we consider a linear subspace $\mathfrak{y}$ of the Lie algebra of all vector fields that contains the vector field $f\partial_x + g\partial_y + h\partial_z$ given by (10) and a Lie algebra $\mathfrak{x}$ of vector fields with $[\mathfrak{x}, \mathfrak{y}] \subseteq \mathfrak{y}$. The latter holds in particular true if $\mathfrak{y} = \mathfrak{x}$ itself is a Lie subalgebra as in the Hamiltonian case (or in the dissipative case when $\mathfrak{y} = \mathfrak{x}$ coincides with the whole Lie algebra of all vector fields). Other examples include volume-preserving vector fields, or vector fields that are equivariant with respect to a symmetry group. In fact, it is also possible to include parameter-dependent vector fields in this way by adding $\dot{\lambda} = 0$ to (10) and considering the Lie algebra $\mathfrak{y}$ of vector fields for which the ∂_λ–component vanishes together with the Lie algebra $\mathfrak{x}$ of vector fields for which the ∂_λ–component depends only on λ.

Arnol'd [1] refined the proof of theorem 1 by applying an ultraviolet cut-off to (6) and thus having to solve only finitely many of the small denominator equations in (7) at every iteration step, with strictly increasing truncation order of the Fourier series during the iteration. To allow for this both $\mathfrak{x}$ and $\mathfrak{y}$ are required to be closed under Fourier truncation. Furthermore, given a (not necessarily integrable) vector field $f\partial_x + g\partial_y + h\partial_z$ and fixed $y_0 \in \mathbb{Y}$ also the dominant part

$$f_\lambda(x, y_0, 0)\partial_x \ + \ (h_\lambda(x, y_0, 0) \ + \ \partial_z h_\lambda(x, y_0, 0) \cdot z)\, \partial_z \tag{12}$$

is required to be in $\mathfrak{x}$ or $\mathfrak{y}$, respectively. For the integrable system (10) this yields the dominant part $\omega_\lambda(y_0)\partial_x + \Omega_\lambda(y_0) \cdot z\, \partial_z$. The vector fields of the form (12) define a linear subspace $\mathfrak{h}$ of $\mathfrak{y}$ and a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{x}$. Finally, the Lie group G_{mat} of the Lie algebra $\mathfrak{g}_{\mathrm{mat}}$ containing the matrices $\Omega_\lambda(y_0)$ that appear in the ∂_z–component of (12) for integrable vector fields in $\mathfrak{g}$ is required to be algebraic. Letting similarly $\mathfrak{h}_{\mathrm{mat}}$ denote the vector space of matrices Ω appearing in $\omega\partial_x + \Omega \cdot z\, \partial_z \in \mathfrak{h}$ we may allow for the huge number $s = n + \dim \mathfrak{h}_{\mathrm{mat}}$ of parameters and simply work with $\lambda = (\omega, \Omega) \in \mathbb{R}^n \times \mathfrak{h}_{\mathrm{mat}}$.

Theorem 5 (Mel'nikov, Moser, Kuksin, Eliasson, Broer, Huitema, Takens) *Let the perturbed vector field*

$$(\omega + f(x, y, z; \omega, \Omega))\, \partial_x \ + \ g(x, y, z; \omega, \Omega)\partial_y \ + \ (\Omega \cdot z + h(x, , y, z; \omega, \Omega))\, \partial_z \tag{13}$$

be real analytic on the closure of the relative compact domain $\mathbb{T}^n \times \mathbb{Y} \times \mathbb{V} \subseteq \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^q$ *with* (ω, Ω) *varying in the closure of a relative compact domain* $\mathbb{O} \subseteq \mathbb{R}^n \times \mathfrak{h}_{\mathrm{mat}}$*. Assume that* $\det \Omega \neq 0$ *for all* $(\omega, \Omega) \in \overline{\mathbb{O}}$*. Then there exists* $\varepsilon > 0$ *such that for all* $\|(f, g, h)\|_\infty < \varepsilon$ *there is a structure-preserving diffeomorphism*

$$\Phi \ : \ \mathbb{T}^n \times \mathbb{Y} \times \mathbb{V} \times \mathbb{O} \ \longrightarrow \ \mathbb{T}^n \times \mathbb{Y} \times \mathbb{V} \times \mathbb{O}$$

near the identity and a measure-theoretically large Cantor set $\mathbb{O}' \subseteq \mathbb{O}$ *with the property that the restriction of* Φ *to* $\mathbb{T}^n \times \mathbb{Y} \times \mathbb{V} \times \mathbb{O}'$ *conjugates* $\omega\partial_x + \Omega \cdot z\, \partial_z$ *with* (13).

One speaks of *quasi-periodic stability* [16, 15] if the conclusion of theorem 5 holds true. Early papers were foremost concerned with exclusively using the y–variables for the necessary control of ω and Ω. Denoting the eigenvalues of Ω by

$$\beta_1 \pm \mathrm{i}\alpha_1, \ldots, \beta_r \pm \mathrm{i}\alpha_r, \delta_{2r+1}, \ldots, \delta_q$$

we have $\delta_j \neq 0$ because $\det \Omega \neq 0$ and $\alpha_j \neq 0$ by convention (otherwise the corresponding β_j would be among the δ_j's). The purely imaginary eigenvalues are the ones with $\beta_j = 0$.

We also write alternatively $\delta_1, \ldots, \delta_q$ for all normal eigenvalues to have a uniform notation when helpful.

During the proof of theorem 5 one has to solve a counterpart of the linearized homological equation (6). Next to the small denominators $2\pi\mathrm{i}\langle k \mid \omega \rangle$ stemming from (7) this leads to additional small denominators as one has to invert the linear operators $2\pi\mathrm{i}\langle k \mid \omega \rangle - \Omega$ and $2\pi\mathrm{i}\langle k \mid \omega \rangle - \operatorname{ad}\Omega$, where $(\operatorname{ad}\Omega)(\Lambda) := [\Omega, \Lambda] = \Omega\Lambda - \Lambda\Omega$ is the adjoint operator and scalars are shorthand notation for scalar multiples of the respective identity mapping. The additional Diophantine conditions

$$\bigwedge_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \quad \bigwedge_{j=1,\ldots,q} \quad |2\pi\mathrm{i}\langle k \mid \omega \rangle \; - \; \delta_j| \;\; \geq \;\; \frac{\gamma}{|k|^\tau} \tag{14}$$

and

$$\bigwedge_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \quad \bigwedge_{j,l=1,\ldots,q} \quad |2\pi\mathrm{i}\langle k \mid \omega \rangle \; - \; (\delta_j - \delta_l)| \;\; \geq \;\; \frac{\gamma}{|k|^\tau} \tag{15}$$

provide the necessary lower bounds and were first formulated by Mel'nikov [34]. The two Mel'nikov conditions (14) and (15) together with (4) are implied[3] by the Diophantine conditions

$$\bigwedge_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \quad \bigwedge_{\substack{\ell \in \mathbb{Z}^r \\ |\ell| \leq 2}} \quad |2\pi\langle k \mid \omega \rangle \; + \; \langle \ell \mid \alpha \rangle| \;\; \geq \;\; \frac{\gamma}{|k|^\tau} \tag{16}$$

which prevent normal-internal resonances between the normal frequencies α_j and the internal frequencies ω_j (and also resonances among the internal frequencies, using $\ell = 0$). Note that the real eigenvalues $\delta_{2r+1}, \ldots, \delta_q$ do not enter (16) and complex eigenvalues $\beta_j \pm \mathrm{i}\alpha_j$ with $\beta_j \neq 0$ do not enter (14). As these hyperbolic eigenvalues can always be split off by means of a centre[4] manifold, cf. [27], we restrict from now on to *elliptic tori* with purely imaginary eigenvalues $\pm\mathrm{i}\alpha_1, \ldots, \pm\mathrm{i}\alpha_r$, so in particular $q = 2r$.

Mel'nikov [34, 35] concentrated on the Hamiltonian case (so $m = n$ and $r = p$) and claimed quasi-periodic stability under Kolmogorov's condition (3) on the (internal) frequency mapping if furthermore

$$\bigwedge_{k \in \mathbb{Z}^n} \quad \bigwedge_{\substack{\ell \in \mathbb{Z}^r \\ |\ell| = 1,2}} \quad 2\pi\langle k \mid \omega(y) \rangle \; + \; \langle \ell \mid \alpha(y) \rangle \;\; \neq \;\; 0$$

for all $y \in \mathbb{Y}$, in particular excluding multiple eigenvalues. Proofs were later given by Kuksin [32, 33], Eliasson [22], Pöschel [40] and Rüssmann [42, 43], using Rüssmann's non-degeneracy condition (9). In fact elliptic tori, for which the *extended frequency mapping*

$$(\omega, \alpha) \; : \;\; \mathbb{Y} \;\; \longrightarrow \;\; \mathbb{R}^n \times \mathbb{R}^r \tag{17}$$

[3] We assume that $\gamma > 0$ is smaller than the smallest absolute value of the non-zero real parts of the eigenvalues.

[4] Note that such a centre manifold does not have to be analytic and may even fail to be of class C^∞. Here the finitely differentiable versions of KAM theory mentioned after the sketched proof of theorem 1 become indespensable.

with $\mathbb{Y} \subseteq \mathbb{R}^n$ can never be a submersion, formed an important motivation for the generalization to (9).

Moser [37] only used time rescaling (*i.e.* (9) with $L = 1$) to confront the lack-of-parameter problem in (17). This allowed him to show quasi-periodic stability of elliptic tori for $r = 1$ and of hyperbolic[5] tori if all eigenvalues are real, both for Hamiltonian systems and for reversible systems (for which he restricted to the classical case $m = n$). Graff [23] and Zehnder [52, 53] then showed persistence of all hyperbolic tori, together with their stable and unstable manifolds.

Broer, Huitema and Takens [29, 16, 14] studied how external parameters $\lambda \in \mathbb{O} \subseteq \mathbb{R}^s$ can be used to exert a form of control on the *amended frequency mapping*

$$(\omega, \Omega) \ : \ \mathbb{Y} \times \mathbb{O} \ \longrightarrow \ \mathbb{R}^n \times \mathfrak{h}_{\mathrm{mat}} \tag{18}$$

that suffices to prove quasi-periodic stability. The key ingredient is the adjoint action

$$\begin{array}{ccc} G_{\mathrm{mat}} \times \mathfrak{h}_{\mathrm{mat}} & \longrightarrow & \mathfrak{h}_{\mathrm{mat}} \\ (T, \Omega) & \mapsto & T \Omega T^{-1} \end{array} \tag{19}$$

which is an embryonic version of how the structure-preserving diffeomorphisms act on the vector fields in $\mathfrak{y}$. The requirement that G_{mat} be an algebraic subgroup of $\mathrm{GL}(q, \mathbb{R})$ ensures that the orbits of (19) are smooth submanifolds of $\mathfrak{h}_{\mathrm{mat}}$, see [21]. The centralizer

$$\ker(\operatorname{ad} \Omega)^T \ = \ \left\{ \Lambda \in \mathfrak{h}_{\mathrm{mat}} \ \middle| \ [\Omega^T, \Lambda] = 0 \right\} \tag{20}$$

is transverse to the orbit $G_{\mathrm{mat}} \cdot \Omega$ of (19). Thus the co-dimension of this orbit within $\mathfrak{h}_{\mathrm{mat}}$ is $c = \dim \ker(\operatorname{ad} \Omega)^T$ and the amended frequency mapping (18) can be composed to

$$(\omega, \pi \circ \Omega) \ : \ \mathbb{Y} \times \mathbb{O} \ \longrightarrow \ \mathbb{R}^n \times \mathbb{R}^c \ , \tag{21}$$

shrinking $\mathbb{O}$ a bit if necessary to obtain a global co-dimension c. Broer, Huitema and Takens [16] required (21) to be submersive, which is equivalent to (18) being transverse to the adjoint orbits. In particular $\Omega_\lambda(y)$ provides a universal unfolding of $\Omega_0 = \Omega_{\lambda_0}(y_0)$. In [16] the case that all eigenvalues of Ω_0 are simple was treated; assuming furthermore that all eigenvalues are purely imaginary we have $c = r$ and can identify (21) with the extended frequency mapping on $\mathbb{Y} \times \mathbb{O}$.

Quasi-periodic stability is rather strong as Diophantine tori not only persist under small perturbations, but also retain their normal linear behaviour. Weakening the assumption of simple non-zero eigenvalues typically also results in a weaker conclusion, see below. However, Xu [48], You [51] and de Jong [30] obtained quasi-periodic stability for lower-dimensional invariant tori in Hamiltonian systems with multiple eigenvalues in $1 : 1 : \cdots : 1$ resonance.

[5] Tori for which all eigenvalues of Ω are hyperbolic, having non-zero real part. Such tori are not normally hyperbolic because of the parametrising y–variable with trivial dynamics. Only their union $\mathbb{T}^n \times \mathbb{Y}$ is a normally hyperbolic manifold.

In Hamiltonian systems the harmonic oscillators (2) are in 1:1 resonance if not only their frequencies coincide, but furthermore the Hessean D^2H is positive (or negative) definite. The indefinite case is called 1:-1 resonance, here the equilibrium is neither a minimum nor a maximum of the Hamiltonian H. In *e.g.* reversible systems there is no distinction between a 1:1 and a 1:-1 resonance, all multiple frequencies posing the same problems.

Ciocci [19, 20], Broer, Hoo and Naudot [28, 12] completely drop the requirement that all eigenvalues be simple, but retain the condition $\det \Omega_0 \neq 0$. The universal unfolding of a matrix Ω_0 with a double pair of purely imaginary eigenvalues contains both matrices where this dissolves to two simple pairs of purely imaginary eigenvalues and matrices where the eigenvalues split off from the imaginary axis. This is reflected in the resulting quasi-periodic stability which still exerts full control on the normal linear behaviour. The details of the quasi-periodic bifurcation accompanying this transition are governed by higher order terms, see [25, 8, 5].

Multiple frequencies appear where the second Mel'nikov condition (15) is not fulfilled. For a purely imaginary eigenvalue $\delta = \mathrm{i}\alpha$ the complex conjugate $\bar{\delta} = -\delta$ is a purely imaginary eigenvalue as well, so (15) is also violated by a resonance

$$2\pi\langle k \mid \omega\rangle \;=\; 2\alpha_j \tag{22}$$

even if the first Mel'nikov condition (14) is satisfied. Passing to a 2:1 covering space by means of co-rotating van der Pol co-ordinates leads to the eigenvalue 0. The lifted vector field is equivariant with respect to the deck group $\{\mathrm{id}, F\}$ of the covering space.

For a simple eigenvalue 0 this situation had already been anticipated in [16]. Let $\mathfrak{g}_{\mathrm{const}}$ and $\mathfrak{h}_{\mathrm{const}}$ denote the constant vector fields of the form $\sigma\partial_z$ in $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and assume[6] that these spaces have the same dimension. Then

$$\begin{array}{rcccl} \mathrm{ad}(\omega_0\partial_x + \Omega_0 \cdot z\,\partial_z) \;:\; & \mathfrak{g}_{\mathrm{const}} & \longrightarrow & \mathfrak{h}_{\mathrm{const}} & \\ & \sigma\partial_z & \longmapsto & -\Omega_0 \cdot \sigma\,\partial_z & = \; [\omega_0\partial_x + \Omega_0 \cdot z\,\partial_z, \sigma\partial_z] \end{array} \tag{23}$$

is invertible if and only if

$$\ker \Omega_0 \,\cap\, \mathfrak{g}_{\mathrm{const}} \;=\; \{0\} \; . \tag{24}$$

Invertibility of (23) ensures that the constant part of any perturbing vector field can be transformed away. Consider now the Lie algebra $\mathfrak{x} = \mathfrak{y}$ of (dissipative) vector fields $f\partial_x + h\partial_z$ on $\mathbb{T}^n \times \mathbb{R}^1$ that are equivariant with respect to the group $\{\mathrm{id}, F\}$ generated by the involution $F(x, z) = (x, -z)$. This enforces $\Omega_0 = \partial_z h(x, 0) \equiv 0$ (even for non-integrable vector fields) but also $\mathfrak{g}_{\mathrm{const}} = \{0\}$ whence (24) is still satisfied.

In [6] this is generalized to multiple zero eigenvalues. The resulting quasi-periodic stability again exerts full control on the normal linear behaviour of the persisting tori. The amended frequency mapping (18) now also provides a universal unfolding of the nilpotent part of Ω_0. In the reversible case $\mathfrak{g}_{\mathrm{const}} = \mathrm{Fix}(R)$ and $\mathfrak{h}_{\mathrm{const}} = \mathrm{Fix}(-R)$ yield persistence of a family of tori undergoing a quasi-periodic pitchfork bifurcation, see [6, 26].

[6] This holds true if $\mathfrak{x} = \mathfrak{y}$ and also in the reversible case, where $\mathfrak{g}_{\mathrm{const}} = \mathrm{Fix}(R)$ and $\mathfrak{h}_{\mathrm{const}} = \mathrm{Fix}(-R)$ both have dimension p.

Bourgain [2, 3], Xu and You [50, 49] not only allow for multiple frequencies, but drop the second Mel'nikov condition (15) altogether (and not restricted to a single $k \in \mathbb{Z}^n$ as in (22)). They still obtain persistence of the tori themselves, but all control on the normal linear behaviour is lost. As exemplified in [9] for resonances (22) the corresponding gap opened by (15) leads to a loss of ellipticity through frequency-halving bifurcations at the boundary and the gap is completely filled by the resulting hyperbolic tori.

4 Bifurcations of tori

The very existence of the invariant tori becomes questionable if the first Mel'nikov condition (14) is dropped. This condition excludes resonances

$$2\pi\langle k \mid \omega \rangle \;=\; \alpha_j \tag{25}$$

which lead in co-rotating van der Pol co-ordinates to the eigenvalue 0 on $\mathbb{T}^n \times \mathbb{Y} \times \mathbb{V}$ itself (and not on a 2:1 covering space) whence (24) might no longer be satisfied. Moser [37] did not require $\det \Omega_0 \neq 0$ and also did not explicitly work with parameters, but postulated modifying terms $\theta \in \mathbb{R}^n$, $\sigma \in \ker \Omega_0^T$ and $\Theta \in \ker(\operatorname{ad} \Omega_0)^T$. These form a vector field

$$\theta\partial_x \;+\; (\sigma + \Theta \cdot z)\partial_z$$

that has to be subtracted from the perturbed vector field (13) to ensure the conjugacy with $\omega_0\partial_x + \Omega_0 \cdot z\,\partial_z$. Note the resemblance with (21): θ provides the variation of the frequency vector ω_0 and $\Omega(\Theta) = \Omega_0 + \Theta$ is a universal unfolding of Ω_0. Furthermore we see that we can restrict to $\sigma \in \ker \Omega_0^T \cap \mathfrak{h}_{\mathrm{const}}$. A non-removable constant vector field $\sigma\partial_z$ pushes the invariant tori away from $\{z = 0\}$ — these may even cease to exist. To clarify the latter point, and also to obtain information on the normal linear behaviour of tori $\mathbb{T}^n \times \{y\} \times \{z\}$ that do exist, one needs conditions on the higher order terms: these have to be non-degenerate, as formulated in [47] using the modifying terms approach. In fact, the non-degeneracy conditions in KAM theory rather correspond to the transversality conditions in bifurcation theory, which is why the following definition is proposed in [10].

Definition 6 (BHT-versality) *A family of vector fields on $\mathbb{T}^n \times \mathbb{Y} \times \mathbb{V}$ with dominant part*

$$\omega_\lambda(y)\partial_x \;+\; (\sigma_\lambda(y) + \Omega_\lambda(y) \cdot z)\,\partial_z \tag{26}$$

is BHT-*versal at $(y, \lambda) = (y_0, \lambda_0) \in \mathbb{Y} \times \mathbb{O}$ if at (y_0, λ_0) the modifying frequency mapping*

$$(\omega, \sigma, \Omega) \;:\; \mathbb{Y} \times \mathbb{O} \;\longrightarrow\; \mathbb{R}^n \times \mathfrak{h}_{\mathrm{const}} \times \mathfrak{h}_{\mathrm{mat}}$$

is transverse to

$$\{\omega_{\lambda_0}(y_0)\} \;\times\; \operatorname{im} \Omega_{\lambda_0}(y_0) \;\times\; G_{\mathrm{mat}} \cdot \Omega_{\lambda_0}(y_0) \;,$$

where $G_{\mathrm{mat}} \cdot \Omega_0$ denotes the orbit of Ω_0 under the adjoint action (19) of G_{mat} on $\mathfrak{h}_{\mathrm{mat}}$.

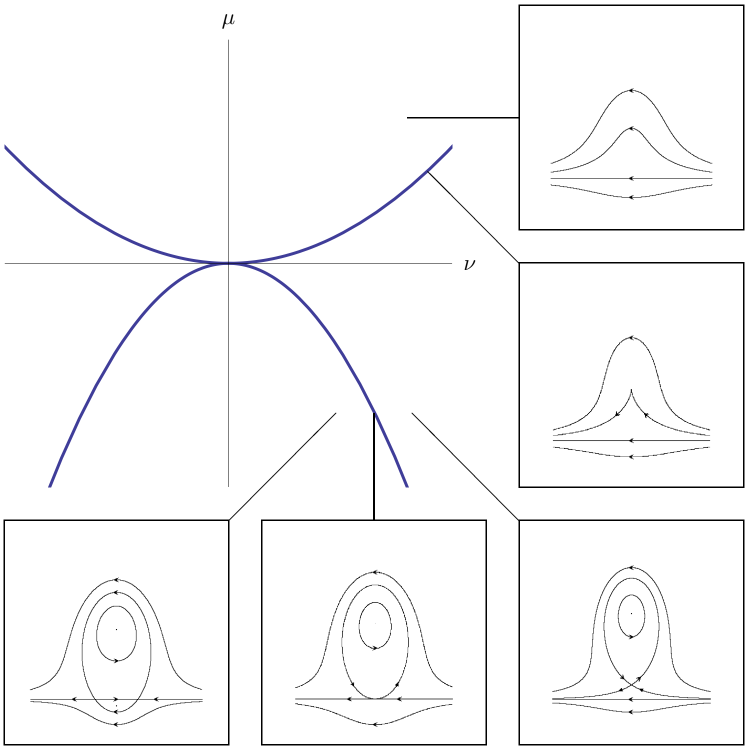

Figure 2: The hyperbolic reversible umbilic bifurcation.

If $\sigma_{\lambda_0}(y_0) = 0$ then $\mathbb{T}^n \times \{y_0\} \times \{0\}$ is an invariant torus of (26) at $\lambda = \lambda_0$. As an example let us consider the vector field

$$\omega_\lambda(y)\partial_x \;+\; z_1 z_2 \partial_{z_1} \;-\; \frac{z_2^2 \pm z_1^2}{2}\partial_{z_2} \tag{27}$$

on $\mathbb{T}^n \times \mathbb{Y} \times \mathbb{R}^2$. Here Ω_0 vanishes completely for the invariant tori $\mathbb{T}^n \times \{y_0\} \times \{0\}$. The ∂_z–component is a Hamiltonian vector field, the Hamiltonian function being the singularity $D_4^\pm$. This is the generic situation for an equilibrium of a one-degree-of-freedom system with linearization $\Omega_0 = 0$, see [4] and references therein. The versal unfolding is given by the hyperbolic umbilic catastrophe for D_4^+ and by the elliptic umbilic catastrophe for D_4^-. Quasi-periodic stability of the resulting bifurcation scenario is proved in [11, 25].

The vector field (27) is also reversible with respect to (11), taking $R(z) = (z_1, -z_2)$. Following [24] this can be used to reduce the number of parameters from 3 to 2 and unfold (27) by

$$\omega_\lambda(y)\partial_x \;+\; (z_1 z_2 + \nu_\lambda(y) z_2)\,\partial_{z_1} \;-\; \frac{1}{2}\left(z_2^2 \;\pm\; (z_1^2 - 2\nu_\lambda(y) z_1) \;+\; 2\mu_\lambda(y)\right)\partial_{z_2} \tag{28}$$

with ∂_z–component undergoing a (hyperbolic or elliptic) reversible umbilic bifurcation. The two bifurcation diagrams are given in figures 2 and 3, see again [24] for more details.

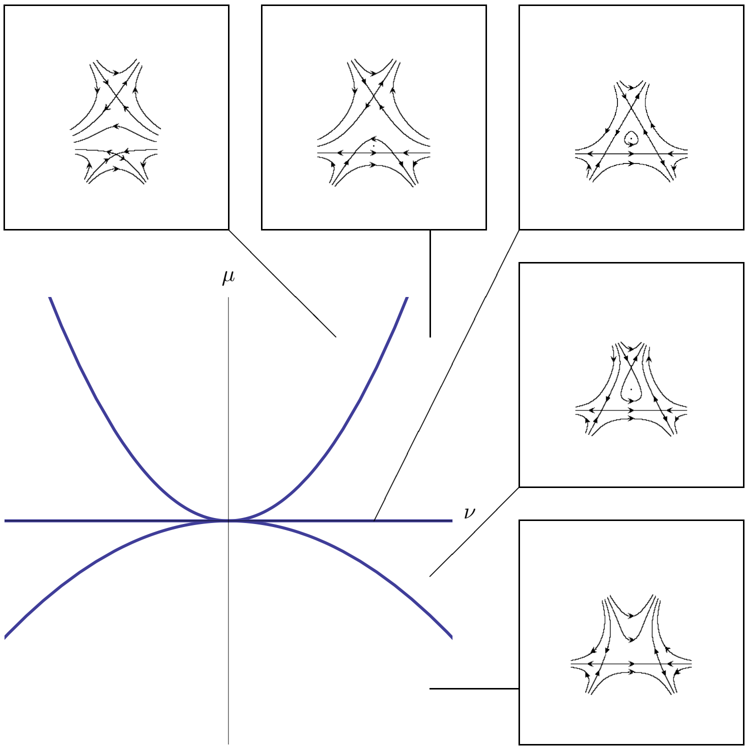

Figure 3: The elliptic reversible umbilic bifurcation.

The family (28) does not satisfy the conditions of definition 6 and to obtain a family that does satisfy these conditions we embed (28) into

$$\omega_\lambda(y)\partial_x \;+\; (z_1 z_2 + \nu_\lambda(y) z_2)\,\partial_{z_1} \;-\; \left(\frac{z_2^2 \pm z_1^2}{2} - \kappa z_1 \;+\; \mu_\lambda(y) \right) \partial_{z_2} \quad . \tag{29}$$

In the present situation $\mathfrak{y}$ denotes the vector space of reversible Hamiltonian vector fields with subspace $\mathfrak{h}$ of dominant parts (12) and $\mathfrak{x}$ denotes the Lie algebra of Hamiltonian vector fields that are equivariant with respect to (11) with Lie subalgebra $\mathfrak{g}$. Thus

$$\mathfrak{h}_{\mathrm{mat}} \;=\; \mathfrak{sp}_-(2,\mathbb{R}) \;=\; \left\{ \Omega \in \mathfrak{sp}(2,\mathbb{R}) \;\middle|\; \Omega R = -R\Omega \right\} \;=\; \left\{ \begin{pmatrix} 0 & \nu \\ \kappa & 0 \end{pmatrix} \;\middle|\; \nu,\kappa \in \mathbb{R} \right\}$$

and

$$\mathfrak{g}_{\mathrm{mat}} \;=\; \mathfrak{sp}_+(2,\mathbb{R}) \;=\; \left\{ \begin{pmatrix} \delta & 0 \\ 0 & -\delta \end{pmatrix} \;\middle|\; \delta \in \mathbb{R} \right\}$$

while $\mathfrak{h}_{\mathrm{const}} = \mathrm{Fix}(-R) = <\partial_{z_2}>$ and $\mathfrak{g}_{\mathrm{const}} = \mathrm{Fix}(R) = <\partial_{z_1}>$. The modifying frequency mapping of (29) reads as

$$(y,\lambda,\kappa) \;\mapsto\; (\omega_\lambda(y), -\mu_\lambda(y)\partial_{z_2}, \begin{pmatrix} 0 & \nu_\lambda(y) \\ \kappa & 0 \end{pmatrix})$$

and is transverse to

$$\{(\omega_{\lambda_0}(y_0), 0, 0)\} \quad \subseteq \quad \mathbb{R}^n \times \mathrm{Fix}(-R) \times \mathfrak{sp}_-(2, \mathbb{R})$$

provided that

$$(\omega, \mu, \nu) \ : \ \ \mathbb{Y} \times \mathbb{O} \quad \longrightarrow \quad \mathbb{R}^{n+2} \tag{30}$$

is a submersion. Given an ε–small reversible Hamiltonian perturbation X of (29), Theorem 3.1 of [47], with the adjustments of [14, 26] to account for reversibility, then yields mappings $\widehat{\omega}, \widehat{\sigma}, \widehat{\Omega}$ satisfying

$$\begin{aligned}
\widehat{\omega}_{\lambda,\kappa}(y,z) &= \omega_\lambda(y) \ + \ \mathcal{O}(\varepsilon) \ \in \ \mathbb{R}^n \\
\widehat{\sigma}_{\lambda,\kappa}(y,z) &= -\mu_\lambda(y) \ - \ \frac{z_2^2 \pm z_1^2}{2} \ + \ \mathcal{O}(z^3) \ + \ \mathcal{O}(\varepsilon) \ \in \ \mathbb{R} \\
\widehat{\Omega}_{\lambda,\kappa}(y,z) &= \begin{pmatrix} 0 & \nu_\lambda(y) \\ \kappa & 0 \end{pmatrix} \ + \ \mathcal{O}(\varepsilon) \ \in \ \mathfrak{sp}_-(2, \mathbb{R})
\end{aligned}$$

and having the following property. The solutions of $\widehat{\sigma}_{\lambda,\kappa}(y,z) = 0$ for which the resulting $(\widehat{\omega}_{\lambda,\kappa}(y), \widehat{\Omega}_{\lambda,\kappa}(y))$ satisfies (16) determine invariant tori of X that have normal linear behaviour conjugate to $\widehat{\omega}_{\lambda,\kappa}(y)\,\partial_x + \widehat{\Omega}_{\lambda,\kappa}(y)\,z\,\partial_z$.

Proposition 7 *If* (30) *is a submersion then the quasi-periodic reversible umbilic bifurcation of* (28) *persists under small reversible Hamiltonian perturbations.*

Sketch of proof. Normalizing the quadratic terms of X allows to turn them back into the form in (27) by scaling z appropriately. Writing $\widehat{\nu}_{\lambda,\kappa}(y) = \nu_\lambda(y) + \mathcal{O}(\varepsilon)$ and $\widehat{\kappa}_{\lambda,\kappa}(y) = \kappa + \mathcal{O}(\varepsilon)$ for the nonzero[7] components of $\widehat{\Omega}_{\lambda,\kappa}(y)$ we aim for $\widehat{\kappa}_{\lambda,\kappa}(y) = \pm\widehat{\nu}_{\lambda,\kappa}(y)$. The translation $z \mapsto (z_1 - \zeta, z_2)$ turns the ∂_z–component of (29) into

$$(z_1 z_2 \ + \ (\nu_\lambda(y) - \zeta) z_2)\,\partial_{z_1} \ - \ \frac{1}{2}\left(z_2^2 \ \pm \ (z_1^2 - 2\zeta z_1 + \zeta^2) \ - \ 2\kappa z_1 \ + \ 2\mu_\lambda(y)\right)\partial_{z_2}$$

whence we choose $\zeta = \frac{1}{2}\left(\widehat{\nu}_{\lambda,\kappa}(y) \pm \widehat{\kappa}_{\lambda,\kappa}(y)\right)$ and make the trivial adjustment $\widehat{\mu}_{\lambda,\kappa}(y) = \mu_\lambda(y) \pm \frac{1}{2}\zeta^2$. □

Alternatively, one can prove this result by dragging the reversing symmetry (11) through the proof in [11].

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[7]Note that the diagonal components of $\widehat{\Omega}_{\lambda,\kappa}(y)$ still vanish because of reversibility.

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