Dynamical stability of quasi-periodic response solutions in planar conservative systems

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Abstract

We study non-autonomous planar Hamiltonian or reversible vector fields that vanish at the origin. The time-dependence is quasi-periodic with strongly non-resonant frequencies. First we give a simple criterion in terms of the averaged system for the trivial solution to be dynamically stable. Then we obtain destabilizations for classes of examples where the conditions of the criterion are not satisfied. We end with possible ways to stabilize an unstable trivial solution by means of vector fields with zero average.

1 Introduction

A Hamiltonian $H = H(q, p)$ for which the linear terms vanish has the origin as an equilibrium. Furthermore we may put the constant term $H(0, 0) = 0$ and in one degree of freedom the low order terms take the form

$$H_{\text{truncated}}(q,p) \quad = \quad \sum_{i+j=2}^{2m} \frac{a_{ij}}{i!\,j!} p^i q^j \quad . \tag{1}$$

The Hamiltonian vector field $X_{H_{\text{truncated}}}$ is reversible with respect to $p \mapsto -p$ if and only if $a_{ij} = 0$ for all odd i. We are interested in quasi-periodically forced vector fields $X_H + Y + Z$ with (fixed) frequencies $\omega_1, \ldots, \omega_n$, turning the equilibrium into an invariant n–torus. To this end we introduce the extended phase space $\mathbb{T}^n \times \mathbb{R}^2$ (with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$) where writing $x = (\omega_1 t, \ldots, \omega_n t) \in \mathbb{T}^n$ for the toral variable yields the desired time-dependence and consider the analytic[1] vector field

$$X_H(q,p) \;+\; Y(x,q,p) \;+\; Z(x,q,p) \quad . \tag{2}$$

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[1] In the periodic case $n = 1$ finite differentiability is enough if some property concerning stability/instability only requires to work to a finite order.

The vector field Z contains the higher order terms, in particular $Z(x,0,0) \equiv 0$ and $DZ(x,0,0) \equiv 0$ (with derivative D in q– and p–directions), while Y has zero average, *e.g.* containing a possible time-dependent counterpart of the coefficients a_{ij} in $X_{H_{\mathrm{truncated}}}$, and also satisfies $Y(x,0,0) \equiv 0$. Since the higher order terms of X_H may be included in Z we drop from now on the suffix 'truncated' and assume that H itself is given by (1). We require (2) to be conservative, *i.e.* either Y and Z are Hamiltonian as well, or, if X_H is reversible, both Y and Z are reversible (but not necessarily Hamiltonian). A first test for dynamical stability of $(q,p) = (0,0)$ is to linearize X_H to

$$\begin{aligned} \dot{q} &= a_{11}q + a_{20}p \\ \dot{p} &= -a_{02}q - a_{11}p \end{aligned} \tag{3}$$

and compute the determinant $a_{20}a_{02} - a_{11}^2$. In the hyperbolic case $a_{20}a_{02} < a_{11}^2$ already the equilibrium of X_H in the time-independent system is unstable. In the elliptic case $a_{20}a_{02} > a_{11}^2$ the equilibrium of X_H is stable, being a centre in one degree of freedom. For the invariant torus $\{(q,p) = (0,0)\}$ of (2) we denote by $\alpha = (\operatorname{sgn} a_{20})\sqrt{a_{20}a_{02} - a_{11}^2}$ the normal frequency and use Bruno conditions

$$\bigwedge_{0 \neq k \in \mathbb{Z}^n} \quad \bigwedge_{\ell \in \{0,\ldots,2m\}} \quad |\langle k \mid \omega \rangle + \ell\alpha| \quad \geq \quad \gamma\Phi(|k|) \tag*{$(4)_{2m}$}$$

(where $\langle .. \mid .. \rangle$ denotes the standard inner product and $|k| = |k_1| + \ldots + |k_n|$) with $\gamma > 0$ and a Rüssmann approximation function $\Phi : [1,\infty[\longrightarrow]0,1]$, a monotonously decreasing function satisfying $\Phi(1) = 1$ and

$$\int_1^\infty \frac{1}{s^2} \ln \frac{1}{\Phi(s)} \, \mathrm{d}s \quad < \quad \infty \ .$$

This generalizes Diophantine conditions where $\Phi(s) = s^{-\tau}$, $\tau > n - 1$ and depending on the choice of Φ the condition becomes either stronger, with better estimates but less frequencies to which these apply, or weaker, with a larger resulting Cantor set of frequencies.

The periodic orbits around a centre are approximated by ellipses, and a rotation to principal axes of the Hessian $\frac{1}{2}D^2H(0)$ together with a scaling turns these into circles, transforming (3) into

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \ .$$

Reduction of the linearization of $X_H + Y$ to this form can be achieved in the periodic case $n = 1$ by means of Floquet's theorem. In the quasi-periodic case $n \geq 2$ reducibility has been obtained in [7, 8] for most sufficiently small Y — writing the linear part of Y as εY_0 there is a large Cantorian subset $\mathcal{E} \subseteq [0,\varepsilon_0]$, $\varepsilon_0 > 0$ for which $DX_H(0) + \varepsilon Y_0$ is reducible. Note that the statement in [7] requires the eigenvalues of the average of $DX_H(0) + \varepsilon Y_0$ to vary with ε from the start in a Lipschitz way (with upper and lower bounds on the derivative) as the normalization scheme will typically produce non-zero averages later on — in the highly exceptional case that throughout the iteration all averages vanish there are no gaps in $\mathcal{E}$. The resulting control of the eigenvalues during the iteration process ensures that the Bruno conditions $(4)_2$ are satisfied throughout the proof, thereby also producing the gaps in $\mathcal{E}$. As discussed in [8] one can alternatively drag the question of non-zero averages through the proof, thereby diminishing the size of the gaps from $\mathcal{O}(\varepsilon)$ to $\mathcal{O}(\varepsilon^m)$ where m designates the iteration step where the first non-zero average occurs. The proof in [7] is written down using Diophantine conditions and immediately generalizes to Bruno conditions. In [8] an exponential bound on the remainder is obtained by using a specially tailored approximation function Φ which is very close to the limit of validity of the Bruno conditions.

From now on we assume that the linear part of (2) is already reduced to constant coefficients; simply requiring that next to $Y(x, 0, 0) \equiv 0$ also $DY(x, 0, 0) \equiv 0$. Making nonlinear terms time-independent is the realm of normal form theory.

Lemma 1 *The Hamiltonian H can be put into Birkhoff normal form*

$$H(q,p) \;=\; \sum_{i=1}^{m} \frac{\alpha_i}{2^i\, i!}(p^2+q^2)^i \;+\; h.o.t. \tag{5}$$

with $\alpha_1 = \alpha \neq 0$.

The proof is standard and proceeds by recursively normalizing the terms of degree $3, 4, \ldots$ up to $2m$. The same procedure can simultaneously remove the time dependence if the internal frequencies $\omega_1, \ldots, \omega_n$ satisfy $(4)_0$ and the normal frequency α is bounded away from normal-internal resonances.

Lemma 2 *For fixed $m \geq 2$ let (ω, α) satisfy $(4)_{2m}$. Then $X_H + Y$ can be put into Birkhoff normal form with a Hamiltonian of the form (5) where $\alpha_1 = \alpha \neq 0$ while $\alpha_2, \ldots, \alpha_m$ may depend on Y.*

This yields stability by a standard application of KAM theory if not all $\alpha_2, \ldots, \alpha_m$ are zero, see [2] and references therein. In fact we choose m to be the first integer ≥ 2 with $\alpha_m \neq 0$ (so in most cases $m = 2$). Then Y contains nonlinear terms up to order $2m - 1$ and all terms of order $2m$ and higher are collected in Z.

Theorem 3 *Let the origin be a centre of X_H and let the normal frequency α together with the internal frequencies of Y and Z satisfy $(4)_{2m}$ for some $m \geq 2$. If $\alpha_m \neq 0$ in the Birkhoff normal form of $X_H + Y$ then the invariant torus $\{(q,p) = (0,0)\}$ of (2) is stable in the sense of Lyapunov.*

In the degenerate[2] case $a_{20}a_{02} = a_{11}^2$ the higher order terms are necessary to determine stability already in one degree of freedom. Indeed, since H is analytic the origin is a stable equilibrium if and only if $H(0, 0)$ is a local extremum. The passage $H \mapsto -H$, which amounts to reversing time, if necessary, allows to restrict to the origin being a minimum. To avoid that already time-independent terms from Z can make the origin

[2]In the literature the name parabolic is used both for the general case of zero eigenvalues [18] and for the special case that furthermore the linearization is not completely zero [5].

unstable we assume this to be a strict minimum. As H is analytic we may restrict to a suitable neighbourhood of the origin to have $H(q,p) > 0$ and $DH(q,p) \neq (0,0)$ for all $(q,p) \neq (0,0)$, cf. [1]. Our main result reads as follows (see theorem 5 in section 3 for a more precise reformulation).

Main Result *Let the origin be a degenerate strict minimum of H and ω satisfy the Bruno conditions* $(4)_0$. *Under suitable conditions on Y and Z the invariant torus $\{(q,p) = (0,0)\}$ of* (2) *is stable in the sense of Lyapunov.*

This result also applies to elliptic minima not satisfying $(4)_2$ because of a normal-internal resonance

$$\langle k \mid \omega \rangle \quad = \quad \ell \alpha \tag{6}$$

with $\ell = 1$ or $\ell = 2$, after passing to an ℓ–fold covering space by means of a van der Pol transformation to co-rotating co-ordinates, cf. [2, 4]. In fact this extends to those cases having zero twist coefficients α_i in (5) where a normal-internal resonance (6) with $\ell = 2m + 1$ or $\ell = 2m + 2$ prevents further normalization.

As the proof of theorem 5 in section 3 shows, a normalization (similar to the one of lemma 2) would allow to remove the purely quasi-periodic part of the coefficients of a time-dependent X_H — in our formulation collected in Y. In fact, not only terms with zero average of the same order as X_H can be removed, but also terms of (a bit) lower order. The order conditions on Y have to ensure that the new terms appearing through the normalization procedure — which no longer need to have zero average — can be collected in the higher order terms together with Z ; we discuss the order of a vector field in section 2. Using an order of Y that is too low we construct in section 4 classes of examples that destabilize the origin $\{(q,p) = (0,0)\}$, and also examples of Y stabilizing an origin that is initially unstable.

In the periodic case $n = 1$ one can alternatively work with a Poincaré mapping, and it is in this form that the first results had been obtained in [17, 18] for the Hamiltonian and then in [14] for the reversible case. In [15] an interpolating Hamiltonian for a given area-preserving Poincaré mapping as treated in [17] is explicitly constructed and the result in [9] is recovered, which is in turn generalized in [10, 11] to quasi-homogeneous H. The paper [16] formulated a stability criterion for elliptic $\{(q,p) = (0,0)\}$, requiring that the first non-zero coefficient $c(t)$ in the nonlinear Hill equation

$$0 \quad = \quad \ddot{q} \; + \; a(t)q \; + \; c(t)q^{2m-1} \; + \; \text{h.o.t.}$$

does not change sign; a similar criterion is given in [15] in the degenerate case.

The first result [3] on the quasi-periodic case again concerned $a_{20} \neq 0$ and was generalized in [12] to semi-quasi-homogeneous H, *i.e.* the sum of a non-degenerate quasi-homogeneous function and a function of strictly larger (weighted) degree, cf. [1, 5]. In the present context the higher order part can be included in Z, the important point is that the quasi-homogeneous H is non-degenerate. In [6] the possibility of certain[3] lower

[3]To prove Lyapunov-stability Y has to be transformed away and next to terms of average zero of the same order as X_H the normalizing-type transformation used to this end also allows to deal with (some) lower order terms of zero average.

order terms with zero average alluded to in [10, 11, 12] was made explicit and furthermore Diophantine conditions were replaced by the Bruno conditions $(4)_0$. For non-degenerate 4th order terms in the normalization of (1) theorems 3 and 5 appeared in [13], where also a version for a reversible but not necessarily Hamiltonian averaged vector field is given.

This paper is organized as follows. In the next section we recall how the order of a vector field is defined when acting on functions that are not semi-quasi-homogeneous. Section 3 contains the formulation and proof of theorem 5 and in the final section 4 we discuss the case of dynamical instability, in particular addressing the question how $\{(q,p) = (0,0)\}$ can be stabilized.

2 Newton Filtrations

In the elliptic case the condition that Y vanishes at the origin makes this time-dependent vector field of the same order as X_H (the higher order terms can always be deferred to Z) and indeed merely makes the coefficients a_{ij} in X_H time-dependent (in the reversible case Y may contain non-Hamiltonian terms of zero average as well). However, as described in [11, 6], in the case of a degenerate linearization $DX_H(0)$ at the origin the quasi-periodic vector field Y may contain terms that are of lower order than X_H. Indeed, the normalization procedure removing a term like $\varepsilon \sin \omega_1 t$ typically results in $\varepsilon^2 \sin^2 \omega_1 t$ which no longer has average zero and therefore must be deferred to Z. One thus expects Y to have higher order than half of the order of X_H, and this is exactly what has been obtained in [6]: there X_H is of order $\delta - 1$, coming from a quasi-homogeneous degree $\delta > 1$ of H, and Y is required to be of order strictly larger than $\frac{1}{2}(\delta - 1)$. Incidentally, $\delta = 1$ corresponds to the elliptic case.

For a general minimum as in theorem 5 the Newton gradation defined by a quasi-homogeneous degree has to be replaced by a Newton filtration, see [20, 1]. To this end we first assign to every pair of coefficients $a_{ij}, a_{kl} \neq 0$ in (1) satisfying $i < k$ and $j > l$ the weights $\beta_p = \frac{j-l}{jk-il}$ and $\beta_q = \frac{k-i}{jk-il}$ and discard the pair if there is a coefficient $a_{mn} \neq 0$ in (1) with $\beta_p m + \beta_q n < 1$ (here m is not necessarily the order in (5) and n is not necessarily the number of frequencies of the quasi-periodic forcing).

The integer points $(j, i) \in \mathbb{N}_0^2$ of the coefficients a_{ij} in the remaining pairs form the vertices of the Newton diagram of H. Because H is analytic and the origin is a strict minimum of H the Newton diagram includes two vertices $(0, k)$ and $(l, 0)$ with both k and l even, see again [1]. In the case that H is semi-quasi-homogeneous, the Newton diagram reduces to the straight line connecting these two points. In general, however, the Newton diagram is a piecewise linear line with edges $\overline{(j_{\mu-1}, i_{\mu-1}), (j_\mu, i_\mu)}$ connecting $(j_0, i_0) = (0, k)$ and $(j_m, i_m) = (l, 0)$ where again the vertices (j_μ, i_μ) must have both co-ordinates i_μ and j_μ even. We denote the weights corresponding to the edges by $\beta^\mu = (\beta_p^\mu, \beta_q^\mu)$, $\mu = 1, \ldots, m$. The Newton diagram thus becomes given by the equation

$$\min_\mu \left(\beta_p^\mu k + \beta_q^\mu l \right) = 1$$

and the Newton filtration is formed by the spaces

$$\mathcal{F}_d := \left\{ K \in C^\omega(\mathbb{R}^2) \;\middle|\; \frac{\partial^{i+j} K}{\partial p^i \partial q^j}(0) = 0 \text{ for all } (j,i) \text{ with } \beta_p^\mu i + \beta_q^\mu j < d \text{ for all } \mu = 1, \ldots, m \right\}.$$

Definition 4 *A vector field $X_x(q,p) = X(x,q,p)$ has order δ if $X_x(\mathcal{F}_d) \subseteq \mathcal{F}_{d+\delta}$ for all $d > 0$.*

In particular if $K \in \mathcal{F}_d$ then X_K has order $d - \beta$ where $\beta = \min_\mu(\beta_p^\mu + \beta_q^\mu)$.

3 Action Angle Variables

Before reformulating (and proving) our main result let us restate the order condition for the case that $Y = X_K$ is Hamiltonian as well, the Hamiltonian function

$$K(x,q,p) \;=\; \sum_{i+j=3}^{2m} \frac{b_{ij}(x)}{i!\,j!} p^i q^j \tag{7}$$

having $\mathbb{T}^n$–dependent coefficient functions b_{ij} with zero average. Then the order of Y is $d - \beta$ where

$$d \;=\; \min_\mu \min_{i,j} \left\{ \beta_p^\mu i \;+\; \beta_q^\mu j \;\middle|\; b_{ij} \neq 0 \right\}$$

with weights β_μ defined by H. The condition formulated below requires $d > \frac{1}{2}(1+\beta)$ and results *e.g.* for the concrete Hamiltonian

$$H(q,p) \;=\; \frac{p^{12}}{12!} + \frac{p^6 q^4}{6!\,4!} + \frac{p^4 q^6}{4!\,6!} + \frac{q^{12}}{12!} \tag{8}$$

in the convex hull of $\{(8,0),(6,1),(1,6),(0,8)\} \subseteq \mathbb{N}_0^2$ to contain the indices (j,i) for which b_{ij} in K may be non-zero, see figure 1.

In particular, for homogeneous H of degree $2m$ the terms in K are required to have a degree strictly higher than $m+1$.

Theorem 5 *Let the origin be a degenerate strict minimum of $H = H(q,p)$ and consider the superposition of the Hamiltonian system with $\dot{x} = \omega$ where ω satisfies the Bruno conditions*

$$\bigwedge_{0 \neq k \in \mathbb{Z}^n} \quad |\langle k \mid \omega \rangle| \;\geq\; \gamma \Phi(|k|) \;.$$

Let $Z = Z(x,q,p)$ be a conservative vector field of order strictly larger than $1-\beta$ with respect to the filtration $(\mathcal{F}_d)_{d>0}$ defined by H (the weights $(\beta_p^\mu, \beta_q^\mu)$ of this filtration determine $\beta = \min_\mu(\beta_p^\mu + \beta_q^\mu)$) and let $Y = Y(x,q,p)$ be a conservative vector field with zero average of order strictly larger than $\frac{1}{2}(1-\beta)$. Then the invariant torus $\mathbb{T}^n \times \{(0,0)\}$ of the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} x \\ q \\ p \end{pmatrix} \;=\; \begin{pmatrix} \omega \\ X_H(q,p) + Y(x,q,p) + Z(x,q,p) \end{pmatrix} \tag{9}$$

is stable in the sense of Lyapunov.

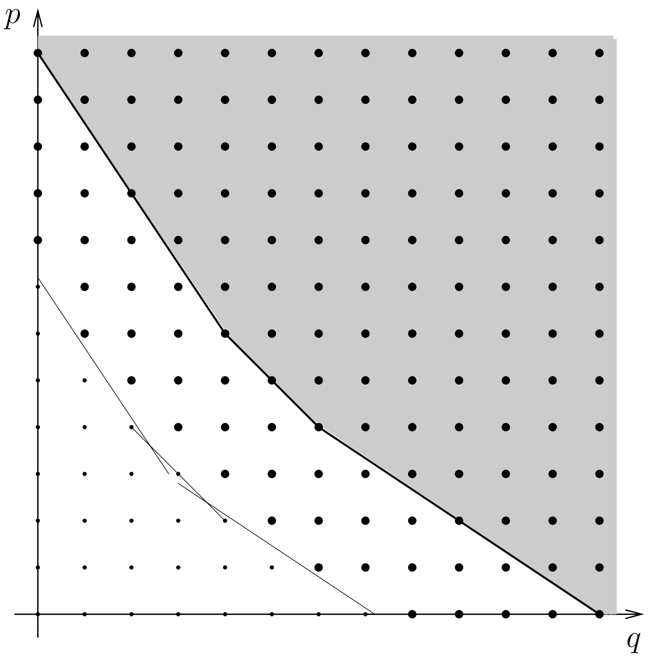

Figure 1: The Newton diagram of (8) separates the grey region of higher order terms from the low order terms. All non-zero coefficients in (7) are required to have indices (j, i) that lie strictly above the shifted boundary $\min_{\mu}(\beta_p^\mu i + \beta_q^\mu j) = \frac{3}{5}$. Note that the coefficient function b_{33} of $p^3 q^3$ has to vanish since the index $(3, 3)$ lies on (and not above) the boundary.

This encompasses the stability results obtained in [17, 18, 9, 10, 11, 3, 12, 6].

Proof. The origin is a strict minimum of H whence all orbits of the autonomous planar system defined by X_H are periodic. Our aim is to use these closed curves to confine the orbits of (9), which would be an immediate application of KAM theory if both Z and Y were of higher order (and thus small as compared to X_H near the origin). To remove the time-dependent lower order terms we follow [6] and use the time–1–mapping ψ_1 of the vector field $-Y$ itself. Since $\frac{\partial}{\partial t}\psi_t = -Y$ cancels Y this results in the transformed vector field

$$X_H \; + \; \dot{q}\frac{\partial}{\partial q}\psi_1 \; + \; \dot{p}\frac{\partial}{\partial p}\psi_1 \; + \; \widetilde{Z} \tag{10}$$

where $\widetilde{Z}$ collects $\psi_1^* Z$, $\psi_1^* X_H - X_H$ and the higher order terms of $\psi_1^* Y$. Action angle variables of the periodic flow defined by X_H are provided by the area A enclosed by an orbit of period τ and the angle $\vartheta = \frac{2\pi t}{\tau}$ of appropriately scaled time. For the indispensable twist condition we follow [3, 12]: a localization $A = \varepsilon + I$ around the closed curve enclosing the area $A = \varepsilon$ followed by a scaling yields the equations of motion

$$\begin{aligned} \dot{I} &= R(t, \varphi, I, \alpha) \\ \dot{\varphi} &= \alpha \; + \; Q(t, \varphi, I, \alpha) \end{aligned}$$

with α encoding the (rescaled) period. The fact that the period tends to infinity as a negative power of the energy level h, when h tends to 0, gives the desired twist condition. To this conservative system Theorems 2.1 and 2.3 of [6] apply, yielding persistence of quasi-periodic solutions and thus confinement of trajectories starting near $\mathbb{T}^n \times (0,0)$. $\square$

The normalizing transformation ψ_1 suffices for our proof. In concrete applications one is also interested in realistic sizes of the neighbourhoods that yield Lyapunov-stability. This can be achieved by further normalization, making terms in $\widetilde{Z}$ independent of time. Where such terms introduce instability the position of the resulting saddle(s) in the truncated normal form allows to estimate the size of the stability domain. As long as all normalized terms re-inforce stability one can continue to make terms in $\widetilde{Z}$ independent of time. Since the vector field (9) is analytic one can in fact normalize so far that the remainder becomes exponentially small in some suitable scaling of the variables.

We now have the following algorithm to study the dynamical stability of a response solution of a quasi-periodically forced planar Hamiltonian system at our disposal. To fix thoughts we first translate the solution to the origin. The first criterion is that the linearization is not hyperbolic.

In the elliptic case we use lemma 2 to normalize the system until $\alpha_m \neq 0$ in (5), yielding stability. If we encounter a normal-internal resonance (6) preventing the desired normalization we pass to an ℓ–fold covering space by means of a van der Pol transformation to co-rotating co-ordinates.

Such a system is also the starting point if the linearization is degenerate, *i.e.* reducible to

$$\begin{pmatrix} 0 & a_{20} \\ 0 & 0 \end{pmatrix} \quad , \quad a_{20} \in \mathbb{R} \ .$$

Here the case $a_{20} = 0$ is included and if $a_{20} < 0$ we can reverse time to achieve $a_{20} > 0$ (changing a possible maximum into a possible minimum). Normalization now amounts to averaging and we proceed until the minimum at the origin has become strict.

Similar considerations apply to a quasi-periodically forced planar reversible system. Recall that in this case the assumption on the averaged lower order terms is more strict — that this vector field be both reversible and Hamiltonian.

4 Destabilizing and Stabilizing

We consider now several classes of examples where theorem 5 does not apply because the assumption on the order of Y is not satisfied. That is, the order of Y is too small compared to that of X_H. We use Hamiltonians K_0 to construct vector fields $Y = X_{K_0}$ with zero $\mathbb{T}^n$–average which destroy the stability of the invariant torus $\{(q,p) = (0,0)\}$ and also look at cases in which the torus is unstable under the flow of X_H and it is made stable by the effect of K_0. In what follows $\mathcal{O}_k$ denotes terms which are of order $\mathcal{O}(\varepsilon^k)$ and given a quasi-periodic function f we shall denote as $\overline{f}$ (resp. $\widetilde{f}$) the average of f (resp. the purely quasi-periodic part of f: $\widetilde{f} = f - \overline{f}$).

Let $H^{(0)} = H(q,p) + \varepsilon K_0(x,q,p)$, where $x \in \mathbb{T}^n$, $\dot{x} = \omega$, be our initial Hamiltonian with K_0 a function that has zero average with respect to x. To recover an autonomous system we introduce the n–dimensional vector of actions y conjugate to x so that the "unperturbed" Hamiltonian is $H(q,p)+\langle\omega,y\rangle$. We look for a canonical transformation to cancel εK_0 that is given by the time–1–mapping of a function $\varepsilon K_1(x,q,p)$, also with zero average. Denoting by $\{..,..\}$ the Poisson bracket, the transformed Hamiltonian is given by

$$\begin{aligned} H^{(1)} &= H + \langle\omega,y\rangle + \varepsilon K_0 + \varepsilon\{H,K_1\} - \varepsilon\langle\omega,\partial K_1/\partial x\rangle + \varepsilon^2\{K_0,K_1\} \\ &+ \frac{1}{2}\left(\varepsilon^2\{\{H,K_1\},K_1\} - \varepsilon^2\{K_0,K_1\} + \varepsilon^3\{\{K_0,K_1\},K_1\}\right) \\ &+ \frac{1}{6}\left(\varepsilon^3\{\{\{H,K_1\},K_1\},K_1\} - \varepsilon^3\{\{K_0,K_1\},K_1\}\right) + \mathcal{O}_4 , \end{aligned} \tag{11}$$

where we have selected K_1 satisfying $\dot{K}_1 := \langle\omega,\partial K_1/\partial x\rangle = K_0$ — this cancels εK_0 in the first line of (11) and is already used in the second and third lines of (11). See also [19] for the general set-up concerning averaging quasi-periodic terms for small arbitrary vector fields, either Hamiltonian or not.

The terms in $\{H,K_1\}$ have zero average, so that they can be cancelled at the next step of averaging which would give us a new Hamiltonian $H^{(2)}$. In a similar way one can cancel the purely quasi-periodic terms which appear in the brackets $\{\cdots\{\{H,K_1\},K_1\},\ldots K_1\}$. Therefore, the most relevant terms are the resulting ε^2–term $\frac{1}{2}\{K_0,K_1\}$ and ε^3–term $\frac{1}{3}\{\{K_0,K_1\},K_1\}$. We shall explore the presence of these terms to study their destabilizing/stabilizing effects.

In what concerns the regularity of K_0, in the general quasi-periodic case we require analyticity in x if we only know that ω satisfies a Bruno condition $(4)_0$. But if ω satisfies a Diophantine condition, choosing $\Phi(s)=s^{-\tau}, \tau>n-1$ in $(4)_0$, then it is sufficient to ask for k continuous derivatives, where k depends on τ and the number of steps of averaging to be applied, so that the coefficients of K_0 decrease in a sufficiently fast polynomial way to ensure that the function K_1 and the successive functions $K_2,\ldots$ introduced by successive averaging steps are quasi-periodic with zero average. In the periodic case $n=1$ it is enough to have K_0 Lebesgue-integrable with respect to t. We apply these ideas to situations of increasing complexity.

Proposition 6 *Consider the Hamiltonian* $H=c_1p^{2l}+c_2q^{2m}$ *with positive coefficients* $c_1,c_2>0$. *Let* $K_0=f_0p^r+g_0q^s$ *where* f_0 *and* g_0 *are periodic or quasi-periodic functions with zero average and the exponents* r *and* s *satisfy* $m(r-1)+l(s-1)<2lm$. *Let* f_1 *and* g_1 *be such that* $\dot{f}_1=f_0$ *and* $\dot{g}_1=g_0$, *both of them with zero average. Assume that the function* $h_1=rs(g_0f_1-f_0g_1)$ *has average* $\overline{h_1}\neq 0$. *Then*

(*i*) *If either* r *or* s *are even, then the origin becomes unstable.*

(*ii*) *If* r *and* s *are odd and* $\overline{h_1}<0$, *then the origin becomes unstable.*

(*iii*) *If* r *and* s *are odd with* $(r-1)s>2l$ *and* $r(s-1)>2m$, *and* $\overline{h_1}>0$, *then the origin remains stable.*

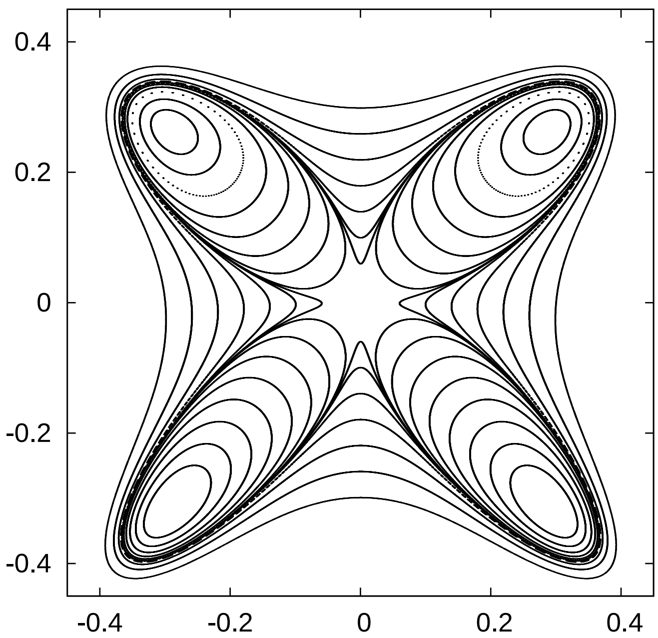

Figure 2: Destabilized origin for $H = \frac{1}{6}p^6 + \frac{1}{6}q^6$ using $K_0 = \frac{1}{3}p^3 \cos t - \frac{1}{3}q^3 \sin t$.

Proof. The condition on r and s ensures that the $p^{r-1}q^{s-1}$–term which appears in $\{K_0, K_1\}$ is below the Newton diagram of H. Clearly the terms in $\{\{H, K_1\}, K_1\}$ contain higher order powers of p and q.

After the first step of averaging the term $\varepsilon\{H, K_1\}$ has zero average and needs another step of averaging to be cancelled. This will require a function, say K_2, which in addition to the terms needed to cancel the purely quasi-periodic part of $\frac{1}{2}\{K_0, K_1\}$ contains the monomials $p^{r-1}q^{2m-1}$ and $p^{2l-1}q^{s-1}$.

After the first step the Newton diagram contains next to $(0, 2l)$ and $(2m, 0)$ from p^{2l} and q^{2m} also $(s-1, r-1)$ from $p^{r-1}q^{s-1}$. Then the assertions (i) and (ii) follow immediately. For (iii) the lower bound on r and s ensures that the $p^{2r-2}q^{s-2}$– and $p^{r-2}q^{2s-2}$–terms which appear in $\{\{K_0, K_1\}, K_1\}$ and similar terms which appear in even higher Poisson brackets do not destabilize the minimum of $c_1p^{2l} + \overline{h_1}p^{r-1}q^{s-1} + c_2q^{2m}$ at the origin. □

For an example of the situation in proposition 6 take $r = s = 3$ and choose periodic functions $f_0 = \frac{1}{3}\cos t$ and $g_0 = -\frac{1}{3}\sin t$, whence $h_1(t) \equiv -1$. Figure 2 shows an illustration using $m = l = 3$, $c_1 = c_2 = \frac{1}{6}$ and $\varepsilon = 0.3$. Then the dominant terms in the Hamiltonian $H^{(2)}$ are $\frac{1}{6}p^6 - \frac{1}{2}\varepsilon^2p^2q^2 + \frac{1}{6}q^6$. Beyond the unstable origin four additional equilibrium points appear at $(q, p) = (\pm\varepsilon, \pm\varepsilon)$ as can be checked in the figure. These points are of elliptic type, with frequencies $2\sqrt{3}\varepsilon^4$, while the origin keeps being degenerate. What is shown are the iterates of the stroboscopic mapping defined by the periodic flow.

A more subtle case of destabilization occurs for $m(r-1) + l(s-1) = 2lm$, whence the point $(s-1, r-1)$ is located on the straight line joining $(2m, 0)$ and $(0, 2l)$. We require that the parameter ε is large enough, simply putting $\varepsilon = 1$, and that h_1 has negative average with sufficiently large modulus. For simplicity we restrict to the homogeneous case $m = l$.

Proposition 7 *Consider the Hamiltonian as in proposition 6 with $l = m \geq 2$, $r = s = m + 1$ and $\overline{h_1} < -2\sqrt{c_1 c_2}/(m+1)^2$. Then the origin is unstable. If m is odd, then the origin is also unstable for $\overline{h_1} > 2\sqrt{c_1 c_2}/(m+1)^2$.*

Proof. The dominant terms after one step of averaging are $c_1 p^{2m} + (m+1)^2 \overline{h_1} q^m p^m + c_2 q^{2m}$. The higher order terms can now have large coefficients, but we are only interested in a vicinity of the origin and, hence, they are irrelevant. The strong negativity of $\overline{h_1}$ implies that the dominant terms are $(\sqrt{c_1} p^m - \sqrt{c_2} q^m)^2$ plus something negative and, hence, change sign. In the case that m is odd the strong positivity of $\overline{h_1}$ makes the Hamiltonian negative on the line $c_1^{\frac{1}{2m}} p + c_2^{\frac{1}{2m}} q = 0$. □

These ideas also work for Hamiltonians H that are not semi-quasi-homogeneous. For instance, in the concrete example (8) the Hamiltonian $K_0 = \frac{1}{6} p^6 \cos t - \frac{1}{6} q^6 \sin t$ still leads to $-\frac{1}{2}\varepsilon^2 p^5 q^5$. As $(5, 5)$ lies on (and not above) the Newton diagram of (8) this results in destabilization as soon as $\varepsilon > \frac{1}{6\sqrt{30}} \approx 0.03043$ (the smallness being due to the large factorial denominators in (8)). Note that the order of $Y = X_{K_0}$ (with respect to the filtration defined by H) coincides with the order $\frac{1}{2}(1 - \frac{1}{5}) = \frac{2}{5}$ given as a boundary in theorem 5 while the monomials p^6 and q^6 lie strictly below the shifted boundary sketched in figure 1.

In special cases this boundary can be improved upon in a systematic way. For the Duffing equation

$$\ddot{q} \; + \; a_0(t) q^l \; + \; c(t) q^{2m-1} \;\; = \;\; 0$$

(where a_0 has zero average and c has positive average) the transformed vector field (10) has the Hamiltonian

$$H(q, p) \;\; = \;\; \frac{p^2}{2} \; + \; a_1(t) \frac{p q^l}{l} \; + \; (a_1(t))^2 \frac{q^{2l}}{2} \; + \; c(t) \frac{q^{2m}}{2m}$$

with $\dot{a_1} = a_0$. Thus, the stability boundary above is shifted from $l = \frac{1}{2}(3m - 1)$ to $l = m$, vindicating a statement in [11, 12].

Recall that theorem 5 yields stability for all coefficient functions $a_0(t)$ with zero average in Duffing's equation. If we restrict to $\mu a_0(t)$ with small μ then the low order term has itself already a stabilizing effect.

Proposition 8 *Consider the equation*

$$\ddot{q} \; - \; \mu a_0(t) \, q^l \;\; = \;\; 0 \; , \quad l \in \mathbb{N}$$

where $a_0 \neq 0$ is an analytic quasi-periodic function having zero average with frequencies $\omega_1, \ldots, \omega_n$ satisfying the Bruno conditions $(4)_0$ and $0 \neq \mu \in \mathbb{R}$. If $l = 1$ we furthermore assume μ to be sufficiently small. Then the trivial solution $q(t) \equiv 0$ is stable in the sense of Lyapunov.

Proof. Changing the sign of a_0, if necessary, we may assume $\mu > 0$ and write $\mu = \varepsilon^2$ with $\varepsilon > 0$. Then the equation can be written as

$$\begin{aligned} \dot{q} &= \varepsilon p \\ \dot{p} &= \varepsilon a_0(t) q^l \ . \end{aligned}$$

We shall carry out several steps of averaging following a different approach, adapting the one in [19]. For the first one we simply change variables by

$$\begin{aligned} x &= q \\ y &= p - \varepsilon a_1(t) q^l \ , \end{aligned}$$

where $\dot{a}_1 = a_0$ with $\overline{a}_1 = 0$, defining a symplectic transformation. We derive the equations for $\dot{x}, \dot{y}$ and immediately rename (x, y) back to (q, p), obtaining

$$\begin{aligned} \dot{q} &= \varepsilon p \ + \ \varepsilon^2 a_1 q^l \\ \dot{p} &= -\varepsilon^2 l a_1 p q^{l-1} \ - \ \varepsilon^3 l a_1^2 q^{2l-1} \ . \end{aligned}$$

To cancel the terms in ε^2 we introduce the transformation

$$\begin{aligned} x &= q \ - \ \varepsilon^2 a_2 q^l \\ y &= p \ + \ \varepsilon^2 l a_2 p q^{l-1} \end{aligned}$$

(where $\dot{a}_2 = a_1$ with $\overline{a}_2 = 0$) which is not symplectic but can be made symplectic by including $\mathcal{O}_4$ terms. As before we derive the equations for these new (x, y) and rename them as (q, p), obtaining

$$\begin{aligned} \dot{q} &= \varepsilon p \ - \ 2\varepsilon^3 l a_2 p q^{l-1} \ + \ \mathcal{O}_4 \\ \dot{p} &= -\varepsilon^3 l \overline{a_1^2} q^{2l-1} \ - \ \varepsilon^3 l \widetilde{a_1^2} q^{2l-1} \ + \ \varepsilon^3 l(l-1) a_2 p^2 q^{l-2} \ + \ \mathcal{O}_4 \ . \end{aligned}$$

Finally the $\mathcal{O}_3$ purely quasi-periodic terms are cancelled by a transformation

$$\begin{aligned} x &= q \ + \ 2\varepsilon^3 l a_3 p q^{l-1} \\ y &= p \ - \ \varepsilon^3 l(l-1) a_3 p^2 q^{l-2} \ + \ \varepsilon^3 l a_4 q^{2l-1} \end{aligned}$$

where $\dot{a}_3 = a_2$ and $\dot{a}_4 = \widetilde{a_1^2}$ with $\overline{a_3} = \overline{a_4} = 0$, a map which differs from symplectic in $\mathcal{O}_6$ terms and which can be made symplectic. The final equations are of the form

$$\begin{aligned} \dot{q} &= \varepsilon p \ + \ \mathcal{O}_4 \\ \dot{p} &= -\varepsilon^3 l \overline{a_1^2} q^{2l-1} \ + \ \mathcal{O}_4 \end{aligned}$$

and have as Hamiltonian

$$\hat{H}(q, p, t) \ = \ \frac{\varepsilon}{2} p^2 \ + \ \frac{\varepsilon^3 \overline{a_1^2}}{2} q^{2l} \ + \ \mathcal{O}_4 \ .$$

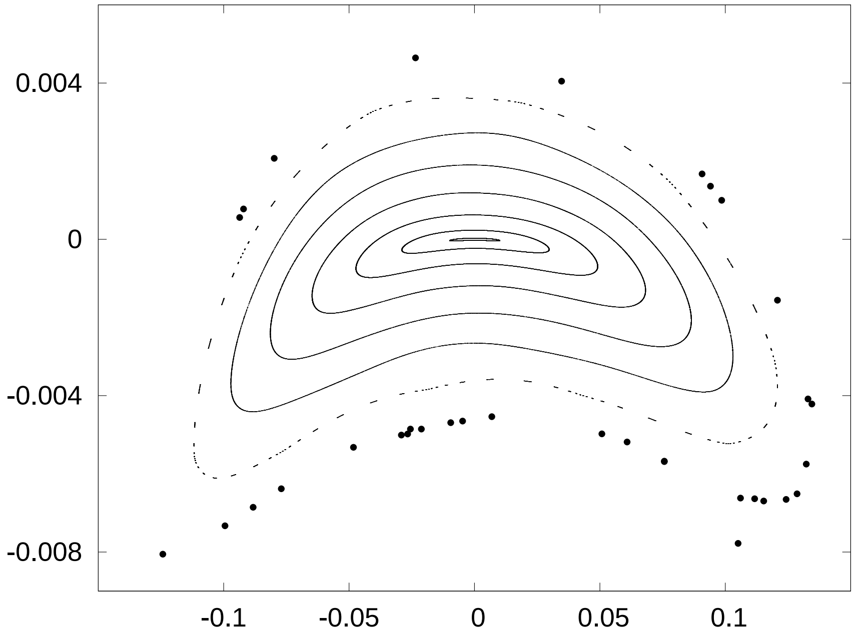

Figure 3: The stabilizing effect of a quasi-periodic perturbation $\frac{1}{15}(\cos x_1 + \cos x_2)q^3$ on the Hamiltonian $H = \frac{1}{2}p^2 - \frac{1}{8}q^8$ with $\omega = (1, \frac{1}{2}(\sqrt{5} - 1))$. Some sliced tori are shown. See the text for details.

For $l = 1$ this immediately shows the stability of the origin; in fact we could go on normalizing until the remainder is exponentially small in ε. For $l \geq 2$ the factor ε^4 in the remainder of the Hamiltonian is not necessarily small, but $\mathcal{O}_4$ is of higher order. Indeed, all remainder terms contain one of the factors p^3, $p^2 q^{2l-2}$, pq^{2l-1} or q^{3l-1} and are therefore above the line passing through $(0, 2)$ and $(2l, 0)$ in the Newton diagram whence we can apply theorem 5 with $Y = 0$ to obtain the desired stability of the origin. $\square$

Thus, for the Hamiltonian

$$H(q, p) \quad = \quad \frac{p^2}{2} - \frac{q^{2m}}{2m} \tag{12}$$

of saddle type we can stabilize the origin $\{(q, p) = (0, 0)\}$ by means of $Y = X_K$, $K = \mu a_0(t) q^{l+1}$ with $1 \leq l < m$ where $a_0 \neq 0$ has zero average (and $\mu \neq 0$ is sufficiently small if $l = 1$).

As a quasi-periodically perturbed example we consider the Hamiltonian $H = \frac{1}{2}p^2 - \frac{1}{8}q^8$ and the perturbation $\frac{1}{3}\mu a_0(t) q^3$ where $a_0(t) = \cos t + \cos \gamma t$, $\gamma = \frac{1}{2}(\sqrt{5} - 1)$. After averaging this term contributes to the Hamiltonian with $\frac{1}{4}(1 + \gamma^{-2})\mu^2 q^4$, giving the desired stabilizing effect. This is analogous to the well-known stabilizing effect of a small and fast enough vibration of the suspension point to stabilize an inverted pendulum. In the present situation the exciting frequency is not fast, but as the dynamics of H is very slow when approaching the origin, it can be considered as a "relatively fast" perturbation and this is enough. Introducing ε was convenient for the method of proof used for proposition 8, but

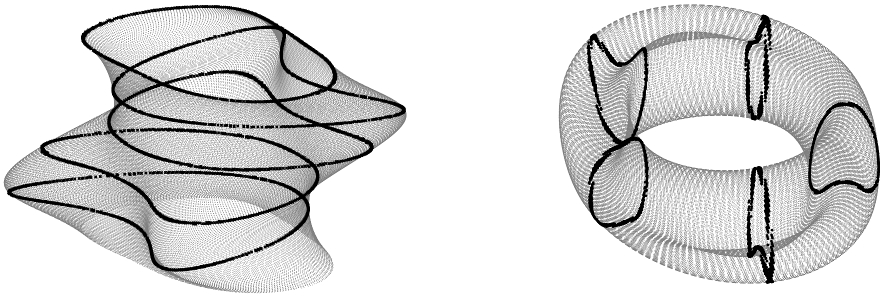

Figure 4: Poincaré-section $x_1 = 0$ for $H = \frac{1}{2}p^2 - \frac{1}{8}q^8 + \frac{1}{15}(\cos x_1 + \cos x_2)q^3$ of the 3–periodic motion with initial conditions $(x_1, x_2, q, p) = (0, 0, 0.09, 0)$. In the left figure the axes are q and p (horizontally) and x_2 (vertically). A slice around $x_2 = x_2^*$ depicted in figure 3 is shown in black and repeated for multiples of $\frac{2\pi}{5}$. Identifying top and bottom yields the more familiar shape to the right.

not essential; and in the presence of q^{2m}–terms that are not multiplied by a power of ε like $\mu = \varepsilon^2$ we certainly avoid doing so. Note that the presence of the negative q^{2m}–term leads to the presence of unstable points at a distance $\mathcal{O}(|\mu|^{1/(m-l)})$ from the origin.

Figure 3 shows some results for $\mu = 0.2$. To represent the dynamics a large set of iterates (hundreds of millions) has been computed using the time–2π–mapping. Then a "slice" has been made by selecting only points $\gamma t \pmod{2\pi}$ within a strip of half width δ around an arbitrary value x_2^*. To be definite $\delta = 10^{-4}$ has been used as half width and $x_2^* = 1.23456789$ as arbitrary value. As expected one can see the closed curves around the origin and also the existence of unstable domains away from the origin. Such instability has to occur eventually because of the term $-\frac{1}{8}q^8$, or due to the presence of additional terms in the normalized Hamiltonian. While about 10^4 points are displayed in every one of the approximate curves, for the last initial point tested only a few points (shown in larger size) are found in the slice before they escape from the frame. The "curves" in figure 3 correspond to invariant 3–tori of the full system on $\mathbb{T}^2 \times \mathbb{R}^2$. For the true time–$2\pi$–mapping (not sliced within $[x_2^* - \frac{\delta}{2}, x_2^* + \frac{\delta}{2}]$) we depict the resulting 2–torus in figure 4, using the initial conditions $(x_1, x_2, q, p) = (0, 0, 0.09, 0)$.

If as in (12) the origin is not an extremum of H, then the Hamiltonian has both positive and negative values in every neighbourhood of the origin and already the equilibrium of X_H in the time-independent system is unstable. Following the lines of the proof of theorem 5 this remains so for the full system (9) if the order conditions stated in theorem 5 are fulfilled; for the quasi-homogeneous Hamiltonians $H = c_1 p^{2l} + c_2 q^{2m}$ with $c_1 c_2 < 0$ more details can be found in [12, 6].

In the periodic case $n = 1$ this yields the well-known instability of periodic orbits with Floquet multiplier $\exp \frac{\mathrm{i}\alpha}{3\omega}$ in a generic system (where the 3rd order terms in (1) do not all vanish) and with Floquet multiplier $\exp \frac{\mathrm{i}\alpha}{4\omega}$ under certain conditions (similar to those of proposition 7) on the 4th order terms in (1). In the quasi-periodic setting these occur at a dense set of α–values. Similar instabilities occur at $\alpha = \frac{1}{\ell}\langle k \mid \omega \rangle$ with higher ℓ in the non-generic case that $\alpha_2 = \alpha_3 = \ldots = 0$ in (5).

Finally we proceed to construct stabilizing low order K_0 (with vanishing average) for Hamiltonians (1) that start with cubic terms. First note that the approach of proposition 6 can also provide a stabilizing result in a situation like case (iii) if in H, beyond the terms of positive coefficients c_1 and c_2 there is a third monomial (*e.g.* $p^l q^m$ with a negative coefficient c_3 with $|c_3|$ sufficiently large). Then H yields an unstable origin but the monomial $p^{r-1}q^{s-1}$ can stabilize it. The following construction can also provide both destabilization and stabilization, depending on the sign of certain averages. Again we restrict to the case of a homogeneous Hamiltonian in proposition 6 and impose the additional condition that $\overline{h_1} = 0$.

Proposition 9 *Consider the Hamiltonian as in proposition 6 with $l = m \geq 2$ and let $K_0 = f_0 p^2 + g_0 q^2$ where f_0, g_0 are periodic or quasi-periodic with zero average. Assume that the function $h_1 = 4(g_0 f_1 - f_0 g_1)$ has zero average.*

(a) If the coefficients c_1, c_2 in H have opposite signs and the averages of $h_1 f_1$ and $h_1 g_1$ have the same sign, then the origin is unstable for $\varepsilon = 0$ and becomes stable for $\varepsilon \neq 0$ (a case of stabilization).

(b) If the coefficients c_1, c_2 have the same sign and the averages of $h_1 f_1$ and $h_1 g_1$ have opposite signs, then the origin is stable for $\varepsilon = 0$ and becomes unstable for $\varepsilon \neq 0$ (a case of destabilization).

Proof. In that case the dominant terms $2h_1(f_1 p^2 - g_1 q^2)$ come from $\{\{K_0, K_1\}, K_1\}$. Under the assumptions on the signs of c_1, c_2 and on the averages of $h_1 f_1$ and $h_1 g_1$ the result is immediate. □

Examples of the cases (a) and (b) in proposition 9 for periodic functions are as follows. Let $f_0 = 2\cos 2t + 3b\cos 3t$ and $g_0 = \cos t$, where b is a free parameter. The different harmonics in f_0 and g_0 immediately yield $\overline{h_1} = 0$. On the other hand $\overline{h_1 f_1} = 3b$ and $\overline{h_1 g_1} = 3$. Hence $b > 0$ gives case (a) and $b < 0$ case (b).

The vector field X_{K_0} of proposition 9 has order 0 (with respect to the homogeneous filtration). Destabilization can also be achieved using a vector field of positive order (lower than the bound $\frac{1}{2}(1-\beta)$ of theorem 5).

Proposition 10 *Let $s > r \geq 2$ be integers and for $i = 0, 1$ define two Hamiltonians K_i on $\mathbb{T}^n \times \mathbb{R}^2$ as*

$$K_i(x, q, p) \;=\; a_i(x)p^r \;+\; b_i(x)p^{r-1}q \;+\; c_i(x)pq^{s-1} \;+\; d_i(x)q^s$$

with quasi-periodic coefficient functions of zero average satisfying $\dot{a}_1 = a_0$, $\dot{b}_1 = b_0$, $\dot{c}_1 = c_0$ and $\dot{d}_1 = d_0$ where $\dot{e}$ abbreviates $\langle \omega \mid \frac{\partial}{\partial x} e \rangle$ and ω satisfies $(4)_0$. Then for the system with Hamiltonian function $\{K_0, K_1\}$ the invariant torus $\mathbb{T}^n \times \{(0,0)\}$ is stable in the sense of Lyapunov if the functions $f_1 = r(b_0 a_1 - a_0 b_1)$, $g_1 = s(d_0 c_1 - c_0 d_1)$ and $h_1 = s(r-1)(d_0 b_1 - b_0 d_1)$ have positive averages $\overline{f_1}, \overline{g_1}, \overline{h_1} > 0$.

Proof. The Poisson bracket $\{K_0, K_1\}$ reads as

$$f_1 p^{2r-2} \; + \; e_1 p^r q^{s-2} \; + \; e_2 p^{r-1} q^{s-1} \; + \; h_1 p^{r-2} q^s \; + \; g_1 q^{2s-2}$$

with $e_1 = r(s-1)(c_0 a_1 - a_0 c_1)$ and $e_2 = rs(d_0 a_1 - a_0 d_1) + (rs - r - s)(c_0 b_1 - b_0 c_1)$. The Hamiltonian function

$$H_1(q,p) \;\; = \;\; \overline{f_1} p^{2r-2} \; + \; \overline{h_1} p^{r-2} q^s \; + \; \overline{g_1} q^{2s-2} \tag{13}$$

has the origin as a strict minimum. Writing the higher order terms of $X_{\{K_0, K_1\}}$ as $Z = X_F$ with $F(x,q,p) = e_1 p^r q^{s-2} + e_2 p^{r-1} q^{s-1}$ and collecting the time-dependent counterpart of (13) in $Y = X_G$ with $G = \{K_0, K_1\} - F - H_1$ we can apply theorem 5. $\square$

An example for a_0, b_0, c_0, d_0 satisfying $\overline{f_1}, \overline{g_1}, \overline{h_1} > 0$ using periodic functions is given by $a_0 = \cos t$, $b_0 = c_0 = \sin t$ and $d_0 = -\cos t$. In case $r > s$ we may simply interchange the variables p and q in proposition 10, but $r = s$ would make the whole function $H_1 + \overline{F}$ in the above proof quasi-homogeneous (in fact even homogeneous) and require assumptions on $\overline{e_1}$ and $\overline{e_2}$ as well.

The unstable origin of a Hamiltonian system defined by H can always be stabilized using proposition 10 by choosing r and s so small that the Newton diagram of (13) lies strictly below the Newton diagram of H. In specific situations it may be possible to increase r and/or s a bit. For instance, if merely a negative coefficient of q^{2m} in H has to be compensated for we can take $s = m$ and even choose $a_0 = b_0 = 0$; the simplest example being $H = p^4 - q^6$, the origin of which is stabilized by $K_0 = pq^2 \sin t - q^3 \cos t$.

Let us stress again that an assumption from the outset was that both Y and Z leave the origin invariant. If the origin is a saddle of X_H then a (sufficiently small) vector field with quasi-periodic coefficients independent of p and q would merely shift the hyperbolic torus $\mathbb{T}^n \times \{(0,0)\}$ a bit. In the elliptic case such a shift already leads to normal-internal resonances Cantorizing the interval $[0, \varepsilon_0]$ of sizes ε of the perturbing vector field. For a degenerate equilibrium at the origin, adding $Y = f(x)\frac{\partial}{\partial q} + g(x)\frac{\partial}{\partial p}$ opens the door to the quasi-periodic bifurcations of [5] and references therein. Concerning our initial problem we leave open the question what happens if we are led to a Bruno condition $(4)_\ell$ that is not satisfied, without α being in normal-internal resonance (6).

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