Functions and Relations

additional reading for the course
Mathematics for AI

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1 Specifying sets

'Set' and 'membership in a set' are basic undefined notions in mathematics. One can develop all of mathematics on the basis of these two basic notions, that is, all other objects encountered in mathematics can be defined in terms of sets. Properties of sets are expressed by the axioms of formal set theory. We do not study them here, but we want to use set-theoretic notation.

Intuitively, a set is any collection of objects of any kind. We can speak about the set of students in the room, the set of stars in the universe, the set of points on the plain, the set of all sets of points on the plain, etc. Objects are unordered and without repetitions, e.g., $\{2, 3, 2, 5\}$ is the same set as $\{5, 3, 2\}$. Sets are called equal, if they have the same elements: $X = Y$ if and only if, for all x, $x \in X \Leftrightarrow x \in Y$.

Sets are usually denoted by capital letters such as X, Y, etc. $x \in X$ can be read 'x is an element of a set X' or 'x is a member of X' or 'x belongs to X'.

Sets can be specified

- By listing all elements, such as $\{\mathrm{red}, \mathrm{green}, \mathrm{blue}\}$. Such sets are necessarily finite.

  Sometimes, large finite or even infinite sets can be specified in a similar manner by using the " ... " notation. Examples: $\{a, \dots, z\}$, $\{0, \dots, 9\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$.

- Using some standard operations. The simplest operations are $\cup$ (union), $\cap$ (intersection), $\setminus$ (difference). We assume the properties of these operations to be known from school. Also recall the so-called *Venn diagrams* to graphically represent these operations.

- By properties. Given a (previously defined) set A the notation

  $$\{x \in A : \varphi(x)\}$$

  specifies the set of all elements x of A that satisfy a property φ.

  Properties can be described in words or using logical notation. In axiomatic set theory $\varphi(x)$ must be a logical formula. We can write $\{x : \varphi(x)\}$ if it is clear from the context which set A the elements of x come from.

  Examples: Both $\{x \in \mathbb{N} : x \text{ is even}\}$ and $\{x \in \mathbb{N} : \exists y \; x = 2y\}$ specify the set of even natural numbers.

  $\{x \in \mathbb{R} : x > 0\}$ specifies the set of positive real numbers.

  $\{x \in A : x \in B\}$ specifies the intersection $A \cap B$.

  $\{x : x \in A \text{ or } x \in B\}$ specifies the union $A \cup B$.

  $\{x \in A : x \notin B\}$ specifies the difference $A \setminus B$.

- One also uses some standard sets which are supposed to be known: $\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), $\varnothing$ (empty set).

Excercise 1 *Write in mathematical notation:*

1. *The set of all vowel letters.*
2. *The set of natural numbers that are squares (of natural numbers).*
3. *The set of non-negative integers. (Specify this set in two different ways.)*
4. *The set of even natural numbers smaller than* 10. *(Specify this set in two different ways.)*
5. *Translate into plain English or Dutch:* $\{x \in \mathbb{Q} : 0 < x < 1\}$.

2 Pairs and cartesian products

Recall that the set $\{a, b\}$ consists of just two elements a and b. We call it an *unordered pair*, because the order in which the elements are listed is not essential. We have:

$$\{a, b\} = \{c, d\} \iff (a = c \text{ and } b = d) \text{ or } (a = d \text{ and } b = c).$$

An *ordered pair* $\langle a, b\rangle$ consists of a and b in the specified order and thus

$$\langle a, b\rangle = \langle c, d\rangle \iff a = c \text{ and } b = d. \qquad (*)$$

In formal set theory one defines $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$, reducing the concept of ordered pair to that of an unordered pair. It is easy to check that $(*)$ is satisfied.

Excercise 2 Prove that this definition works: If $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$, then $a = c$ and $b = d$.

Cartesian product of A and B is defined as

$$A \times B = \{\langle a, b\rangle : a \in A,\ b \in B\}.$$

The set $A \times A \times \cdots \times A$ (n times) is denoted A^n.

Example 1 $\{0, 1\} \times \{2, 3\} = \{\langle 0, 2\rangle, \langle 0, 3\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle\}$.

Example 2 $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the set of ordered pairs of real numbers. They can be interpreted as coordinates of the points on the plain. In mathematics one often simply says that $\mathbb{R}^2$ *is* the set of points on the plain.

Example 3 An integer is essentially a pair $\langle \epsilon, n\rangle$, where $n \in \mathbb{N}$ and $\epsilon \in \{+, -\}$. For example, we think about the number -6 as the pair $\langle -, 6\rangle$. So, integers can be formally defined by

$$\mathbb{Z} = \{0\} \cup (\{+, -\} \times \mathbb{N}).$$

Excercise 3 Let $\rho(x, y)$ denote the distance between the points x and y on the plain. Specify: the set of points on the plain that have equal distance from the the points $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$.

Excercise 4 Write out all elements of the set $\{0, 1\} \times \{0, 1, 2\}$.

Excercise 5 What is the intersection of the following two sets:

$$\{\langle x, y\rangle \in \mathbb{R}^2 : x + y = 1\} \cap \{\langle x, y\rangle \in \mathbb{R}^2 : 2x + 3y = 2\}?$$

3 Functions

A *function* $f : A \to B$ is a subset $f \subseteq A \times B$ such that for each $x \in A$ there is *exactly one* $y \in B$ such that $\langle x, y \rangle \in f$. One usually writes $f(x) = y$ instead of $\langle x, y \rangle \in f$. One usually thinks about a function as a rule assigning to every $x \in A$ a unique $y \in B$. However, we do not need a new basic notion 'rule', but explicate it in terms of sets. Synonyms for the word 'function' are: *mapping* or *map*.

Example 4 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is the same thing as the set of pairs $\{\langle x, y \rangle \in \mathbb{R}^2 : y = x^2\}$. Notice that this set is also called the *graph* of f.

Example 5 For every set A we have the *identity function* $id_A : A \to A$ such that $id_A = \{\langle x, x \rangle : x \in A\}$. In other words, $id_A(x) = x$, for all $x \in A$.

The given definition of function formally reduces the concept of function to the concept of set. This is somewhat contrary to the common intuition, because we are used to think about functions *intensionally*, that is, as laws specifying $y \in B$ for every $x \in A$. Intuitively, functions also have some *dynamic* aspect: the change from x to y. The present definition ignores these aspects. It emphasizes that any thinkable correspondence uniquely associating y with x defines a function.

The set of all functions $f : A \to B$ is denoted B^A.

Example 6 $\{0, 1\}^{\mathbb{N}}$ is the set of all functions $f : \mathbb{N} \to \{0, 1\}$.

3.1 Domain and range

Given a function $f : A \to B$, we say that A is the *domain* of f. It is denoted $\mathrm{dom}(f)$.

The set $\{f(x) : x \in A\} = \{y \in B : \exists x \in A\ f(x) = y\}$, is called the *range* of f. It is denoted $\mathrm{rng}(f)$.

In general, if $X \subseteq A$, then the set $\{f(x) : x \in X\}$ is denoted $f(X)$. If $Y \subseteq B$, then the set $\{x \in A : f(x) \in Y\}$ is denoted $f^{-1}(Y)$.

Excercise 6 Verify that for any function $f : A \to B$ and $X \subseteq A$ one has $X \subseteq f^{-1}(f(X))$. Give an example of a function f and a set $X \subseteq A$ such that $f^{-1}(f(X)) \neq X$.

Excercise 7 Verify that for any function $f : A \to B$ and $Y \subseteq B$ one has $f(f^{-1}(Y)) \subseteq Y$.

3.2 Specifying functions

To specify a function f means to specify A, B and a set of pairs, a subset of $A \times B$. Functions can be defined in the following ways.

- By formulas. For example, a function $f : \mathbb{R}^2 \to \mathbb{R}$ can be defined by $f(x, y) = x^2 + y^2$. If one wants to be quite formal[1], this means that f is the set of pairs of the form $\langle\langle x, y\rangle, x^2 + y^2\rangle$ for $x, y \in \mathbb{R}$.

  Notice that $\mathrm{rng}(f) = \{x \in \mathbb{R} : x \geq 0\}$, because every non-negative number is a square. $\mathrm{dom}(f) = \mathbb{R}^2$.

- By more complex instructions, such as

$$f(x) = \begin{cases} x^2, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

  Checking that such an instruction correctly defines a function usually requires a proof of two things: 1) that a value y is associated to *every* argument $x \in A$; 2) that *only one* such value is associated.

- Finite functions can be specified by finite tables of values, for example a function $f : \{0, 1, 2\} \to \{0, 1\}$ can be specified by the following table:

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

  This means that $f(0) = 0$, $f(1) = 1$ and $f(2) = 0$.

3.3 Composition of functions

Let $f : A \to B$ and $g : B \to C$. *Composition* of f and g is the function $h : A \to C$ defined by $h(x) = g(f(x))$, for any $x \in A$. We denote $h = g \circ f$, so that

$$(g \circ f)(x) = g(f(x)) \tag{1}$$

Notice that the composition $g \circ f$ is only defined if $\mathrm{rng}(f) \subseteq \mathrm{dom}(g)$.

Theorem 1 *The composition operation $\circ$ is associative:*

$$f \circ (g \circ h) = (f \circ g) \circ h,$$

whenever these expressions are defined.

[1] which one never does

Proof. Let $h : A \to B$, $g : B \to C$ and $f : C \to D$. We have to prove that, for each $x \in A$,

$$(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x).$$

Consider an arbitrary $x \in A$. By the definition of composition (1) we obtain:

$$\begin{aligned} (f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\ &= f(g(h(x))) \\ &= (f \circ g)(h(x)) \\ &= ((f \circ g) \circ h)(x). \end{aligned}$$

Since this holds for all $x \in A$, this proves the claim. ⊠

Example 7 (Advanced) Consider the set of all functions from A to A, that is, A^A. Composition operation can be thought of as function from $A^A \times A^A$ to A^A. Indeed, composition associates with every pair of functions $f, g : A \to A$ a unique function $(f \circ g) : A \to A$. So, composition is a function of functions.

3.4 Injective and surjective functions

A function $f : A \to B$ is *injective*, if for all $x, y \in A$,

$$x \neq y \Rightarrow f(x) \neq f(y).$$

A function $f : A \to B$ is *surjective*, if $\mathrm{rng}(f) = B$, that is, every element $y \in B$ equals $f(x)$ for some $x \in A$.

Example 8 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2^x$ is injective, but not surjective. Indeed, $2^x > 0$ for any x, so $0 \notin \mathrm{rng}(f)$. On the other hand, if $x < y$, then $2^x < 2^y$, therefore f is injective.

Example 9 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither injective, nor surjective. Indeed, $-1 \notin \mathrm{rng}(f)$ and $f(-1) = f(1) = 1$ contradicting injectivity.

Excercise 8 What is the set $f^{-1}(\{y\})$ for $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$? Consider separately positive and negative y.

Excercise 9 (Advanced) The same question for the function $f : \mathbb{C} \to \mathbb{C}$ given by the same formula $f(x) = x^2$. Is this function a) injective, b) surjective?

A function is *bijective* if it is both surjective and injective.

Example 10 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 3$ is bijective.

Example 11 The function $f : \{x \in \mathbb{R} : x > 0\} \to \mathbb{R}$ given by $f(x) = \log_2(x)$ is bijective.

Theorem 2 *A function $f : A \to B$ is bijective if and only if there is a function $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$.*

Such a function is called the *inverse of* f and denoted f^{-1}.

Proof. We have to prove two statements.

1. If f is bijective, then there is an inverse function g.
2. If f has an inverse, then f is bijective.

Proof of 1. Suppose f is bijective. We have to construct the inverse function g. Recall that $f \subseteq A \times B$ is a set of pairs $\langle x, y\rangle$. Define a subset $g \subseteq B \times A$ as follows:

$$g = \{\langle y, x\rangle : \langle x, y\rangle \in f\}.$$

We have to check that $g : B \to A$ is a function and that g is, indeed, the inverse of f.

Since f is surjective, for each $y \in B$ there is an $x \in A$ such that $f(x) = y$. Then $\langle x, y\rangle \in f$ and $\langle y, x\rangle \in g$. So, $g(y) = x$ is defined.

Assume $\langle y, x_1\rangle \in g$ and $\langle y, x_2\rangle \in g$, then $f(x_1) = y$ and $f(x_2) = y$. By injectivity of f we obtain $x_1 = x_2$. Hence, the value $g(y)$ is uniquely defined. So, the subset g is a function.

Excercise 10 Prove that $g \circ f = id_A$ and $f \circ g = id_B$.

Proof of 2. Assume that $g : B \to A$ is an inverse of f. We must prove that f is surjective and injective.

Injectivity. If $f(x_1) = y$ and $f(x_2) = y$, then $g(f(x_1)) = g(y) = g(f(x_2))$. However, $g(f(x_1)) = (g \circ f)(x_1) = id_A(x_1) = x_1$. Similarly, $g(f(x_2)) = x_2$, so $x_1 = x_2$.

Surjectivity. We must show that, for each $y \in B$, there is an $x \in A$ such that $f(x) = y$. Consider $x = g(y)$. We have: $f(x) = f(g(y)) = id_B(y) = y$. Hence, x is as required. ⊠

Excercise 11 What is the inverse function to the one defined in Example 11?

Excercise 12 Prove that composition of injective functions $f : A \to B$ and $g : B \to C$ is injective.

Excercise 13 Prove that, if $f : A \to B$ is bijective, then $f^{-1} : B \to A$ is also bijective.

Excercise 14 Describe all injective functions $f : \{0,1\} \to \{0,1,2\}$ by tables of values.

Excercise 15 Describe all bijective functions $f : \{0,1,2\} \to \{0,1,2\}$ by tables of values.

Excercise 16 Prove that, if $f : B \to C$ and $g : A \to B$ are bijective functions, then $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ (in that order!).

3.5 Finite sets

A set A is called *finite*, if there is a bijection $f : \{1, \dots, n\} \to A$. The number n is then called the number of elements of A. It is denoted $n(A)$ or $|A|$.

To be sure that $n(A)$ is uniquely defined, one should prove that if there is a bijection $f : \{1, \dots, n\} \to A$, then there is no bijection $f : \{1, \dots, m\} \to A$ for any other $m \neq n$. This will follow from Theorem 4 below.

The following statement is sometimes taken as an axiom. It can be proved by induction, though.

Theorem 3 (Pigeonhole Principle) *There is no injective function*

$$f : \{1, \dots, n+1\} \to \{1, \dots, n\}.$$

¿From this principle we obtain a corollary.

Theorem 4 *Let A and B be finite sets. There is a bijection between A and B if and only if $n(A) = n(B)$.*

Proof. If $n(A) = n(B) = n$, then there are bijections $f : \{1, \dots, n\} \to A$ and $g : \{1, \dots, n\} \to B$. Then $g \circ f^{-1}$ is a bijection from A to B.

Assume $n(A) = k$ and $n(B) = m$ for some $k < m$. There are bijections $f : \{1, \dots, k\} \to A$ and $g : \{1, \dots, m\} \to B$. If there is a bijection $h : B \to A$, then $h_1 = g \circ h \circ f^{-1}$ is a bijection from $\{1, \dots, m\}$ to $\{1, \dots, k\}$. The restriction of h_1 to $\{1, \dots, k+1\}$ is injective, contradicting the Pigeonhole Principle. ⊠

3.6 Sequences and real numbers

A function $f : \mathbb{N} \to A$ is called an (infinite) *sequence* of elements of A. One often represents sequences as lists $a_1, a_2, a_3, \ldots$ corresponding to values $f(1) = a_1, f(2) = a_2, \ldots$.

Example 12 We can understand a real number as a pair $\langle m, f \rangle$, where m is an integer and f is a sequence of decimal digits. For example, we consider the number $-3,14159\ldots$ as a pair $\langle -3, f \rangle$, where f is the sequence $1, 4, 1, 5, 9, \ldots$.

A sequence of digits $x_1, x_2, \ldots$ is just a function $f : \mathbb{N} \to \{0, \ldots, 9\}$. But there is a condition that period 9 is forbidden, so there must be infinitely many n such that $f(n) \neq 9$. Let I be the set of all such sequences, formally

$$I = \{f \in \{0, \ldots, 9\}^{\mathbb{N}} : f(n) \neq 9 \textit{ for infinitely many } n\}.$$

Then $\mathbb{R}$ can be defined as $\mathbb{Z} \times I$.

3.7 Countable and uncountable sets

A set A is called *countable*, if there is a surjective function $f : \mathbb{N} \to A$.

Theorem 5 *A is countable if and only if A is finite or there is a bijection $f : \mathbb{N} \to A$.*

Proof. Let $f : \mathbb{N} \to A$ be surjective. Consider the set

$$X = \{n \in \mathbb{N} : f(n) \notin \{f(1), \ldots, f(n-1)\}\}.$$

(We also postulate that $1 \in X$.)

We claim: $f : X \to A$ is bijective.

Indeed, if $x, y \in X$ and (say) $x < y$, then $f(y) \notin \{f(1), \ldots, f(y-1)\}$, whereas $f(x) \in \{f(1), \ldots, f(y-1)\}$. So, $f(x) \neq f(y)$. Therefore f is injective.

To establish $f(X) = A$ we prove by induction on n that, for all $n \in \mathbb{N}$, $f(n) \in f(X)$.

For $n = 1$ this follows from $1 \in X$. Assume now $f(1), \ldots, f(n) \in X$. Either $f(n+1) \in \{f(1), \ldots, f(n)\}$ and then, by the induction hypothesis, $f(n+1) \in f(X)$. Or $f(n+1) \notin \{f(1), \ldots, f(n)\}$, and then $n+1 \in X$, so $f(n+1) \in f(X)$, too.

Knowing that $f : X \to A$ is bijective we conclude that, if X is finite, then A is finite.

If X is infinite, then let $g(n)$ be the n-th element in X. Then $g : \mathbb{N} \to X$ is bijective, so the composition $f \circ g$ is a bijection from $\mathbb{N}$ to A. ⊠

Thus, all infinite countable sets are in a bijective correspondence with $\mathbb{N}$.

Theorem 6 *If A and B are countable, then so is the set $A \times B$.*

Proof. Let $a_1, a_2, \ldots$ be an enumeration of A, and $b_1, b_2, \ldots$ an enumeration of B. We can arrange pairs of elements $\langle a_n, b_m \rangle$ in the following infinite list.

$$\langle a_1, b_1 \rangle, \langle a_1, b_2 \rangle, \langle a_2, b_1 \rangle, \langle a_3, b_1 \rangle, \ldots$$

Having enumerated all pairs $\langle a_n, b_m \rangle$ such that $n+m = 2$ and $n+m = 3$, we proceed with those corresponding to $n + m = 4$, $n + m = 5$, etc. Let $g(n)$ denote the n-th pair in this list. Then $g : \mathbb{N} \to A \times B$ is a surjective function. ⊠

Are all infinite sets countable? Georg Cantor answered this by the following famous theorem.

Theorem 7 (Cantor) *The set $\mathbb{R}$ is not countable.*

Proof. Assume $f : \mathbb{N} \to \mathbb{R}$ is a surjective function. Consider the real number whose decimal decomposition (an infinite sequence of digits) $0, x_1 x_2 \ldots$ is defined as follows:

$$x_n = \begin{cases} 0, & \text{if } n\text{-th digit of } f(n) \text{ after the comma is not } 0 \\ 1, & \text{otherwise.} \end{cases}$$

Then we see that the number x differs from each number $f(n)$ in the n-th digit. Therefore, x does not appear in the sequence f, that is, $x \notin \mathrm{rng}(f)$. So, f cannot be surjective contrary to the hypothesis. ⊠

3.8 Cantor's Continuum Hypothesis

Cantor formulated the following *Continuum Hypothesis*: for every subset $A \subseteq \mathbb{R}$, either A is countable, or there is a bijection $f : A \to \mathbb{R}$.

This was a famous open problem in mathematics for many years. Finally, P. Cohen in 1961 proved that this hypothesis cannot be proved on the basis of the axioms of set theory. Earlier K. Gödel proved that this statement cannot be refuted, either. Thus, we do not really know if it is true or false. Statements that can neither be proved nor refuted are called *independent*.

4 Binary relations

Binary relation R on a set A is any subset $R \subseteq A \times A$. One often writes xRy instead of $\langle x, y \rangle \in R$.

Example 13 $<$ is a binary relation on $\mathbb{N}$.

Example 14 $R = \{\langle x, y \rangle$: *x and y are words of the same length*$\}$ is a binary relation on the set of English words.

Example 15 $R = \{\langle x, y \rangle$: *x and y are bus stops in Utrecht and one can reach y from x without changing a bus*$\}$ is a binary relation on the set of bus stops in Utrecht.

A binary relation R on A is *transitive*, if for all $x, y, z \in A$,

$$xRy \text{ and } yRz \Rightarrow xRz.$$

R is *symmetric*, if for all $x, y \in A$,

$$xRy \Rightarrow yRx.$$

R is *reflexive*, if xRx, for all $x \in A$.
R is an *equivalence relation*, if it is reflexive, symmetric and transitive.

5 Equivalence relations

Equivalence relations are widely used to construct more complex sets.

Let R be an equivalence relation on A and let $a \in A$. The *equivalence class of* a is the subset $a_R = \{x \in A : aRx\}$ of A. We have the following properties:

- $a \in a_R$.
- If aRb, then $a_R = b_R$. Indeed, if bRx, then aRx by transitivity. If aRx, then bRa by symmetry and hence bRx by transitivity.
- If not aRb, then $a_R \cap b_R = \varnothing$. Indeed, if $x \in a_R \cap b_R$, then aRx and bRx. Hence xRb by symmetry and aRb by transitivity, a contradiction.

Thus, we have proved the following theorem.

Theorem 8 *An equivalence relation on A defines a* partition *of A into non-intersecting subsets (equivalence classes).*

The set of equivalence classes $\{a_R : a \in A\}$ is denoted A/R.

Example 16 A rational number $q = \frac{m}{n}$ can be considered as a pair $\langle m, n \rangle$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. However, some pairs represent the same number q. Therefore, we define an equivalence relation R on $\mathbb{Z} \times \mathbb{N}$ by

$$\langle m_1, n_1 \rangle R \langle m_2, n_2 \rangle \iff m_1 n_2 = n_1 m_2.$$

(Verify that R is indeed an equivalence relation!)

Rational numbers q can be identified with the equivalence classes. Thus, $(\mathbb{Z} \times \mathbb{N})/R$ is the official definition of the set of rational numbers $\mathbb{Q}$.

Excercise 17 Let S be the set of all words in the alphabet $\{a, b\}$ of length 3. Let

$$R = \{\langle u, v \rangle : u, v \in S \text{ and } u, v \text{ have equal length}\}.$$

Check that R is an equivalence relation on S and describe all equivalence classes.

Excercise 18 Let $f : A \to B$ be a function. Show that

$$R = \{\langle x, y \rangle : x, y \in A \text{ and } f(x) = f(y)\}$$

is an equivalence relation on A.

Excercise 19 Show that in the previous exercise there is a bijection between A/R and B.

5.1 Orders

A binary relation R is called a *(partial) order*, if it is transitive, reflexive and *antisymmetric*, that is,

$$xRy \text{ and } yRx \Rightarrow x = y.$$

In case R is a partial order and xRy, we say that x is a *predecessor* of y and y is a *successor* of x.

More formally, a partial order is a pair $\langle A, R \rangle$, where A is a set and R is a binary relation on A satisfying the above properties. Explanatory example.

Example 17 The relation $\leq$ is a partial order on $\mathbb{N}$. It is also a partial order on $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$. $\langle \mathbb{N}, \leq \rangle$, $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{R}, \leq \rangle$ are, however, all different orders.

These orders are also *linear*, that is, they satisfy the condition

$$xRy \text{ or } yRx,$$

for each $x, y \in A$.

Here is an example of a non-linear partial order.

Example 18 The relation $\subseteq$ is a partial order on set $\mathcal{P}(A)$ of all subsets of A.

Excercise 20 Give an example demonstrating that the ordering $\langle \mathcal{P}(A), \subseteq \rangle$ is non-linear if $n(A) > 1$.

Partial orders can be represented by oriented graphs, where edges connect all elements $x, y \in A$ satisfying xRy. (One usually omits the edges that can be inferred from the transitivity.)

Excercise 21 Draw a diagram representing $\mathcal{P}(\{0, 1, 2\})$ ordered by $\subseteq$.

Excercise 22 Assume that R is a transitive and reflexive relation on A. Show that the relation $\bar{R}$ defined by

$$x\bar{R}y \iff (xRy \text{ and } yRx)$$

is an equivalence relation.

Equivalence classes with respect to $\bar{R}$ are sometimes called 'clusters'. They consist of groups elements that are pairwise related to each other by R. Notice that a binary relation, in which all clusters are one-element sets, is antisymmetric. In fact, the antisymmetry property simply *says* that every cluster is trivial. This is the hint to the following exercise.

Excercise 23 Assume R is as in the previous exercise. Define a binary relation on $A/\bar{R}$ by

$$XQY \iff \exists x \in X \, \exists y \in Y \; xRy.$$

(Here, $X, Y \in A/\bar{R}$ are $\bar{R}$ equivalence classes.)

Show that Q is a partial order.

A partial order R is a *tree-like*, if it satisfies

$$xRa \text{ and } yRa \Rightarrow xRy \text{ or } yRx.$$

In other words, R is tree-like, if the set of predecessors of any element is linearly ordered by R.

Example 19 Any linear order is tree-like.

Example 20 Consider the set of all words in the alphabet A ordered by the relation R such that

$$xRy \iff x \text{ is an initial segment of } y.$$

For example, $x = abc$ is an initial segment of $abcab$.

Then R is tree-like.

Excercise 24 Draw the diagrams of *all* tree-like orders on $\{0, 1, 2\}$.

Now we introduce an important notion of isomorphism. Informally, two mathematical objects are called isomorphic if they 'have the same structure'. For the case of orders we can give the following formal definition.

The orders $\langle A, R_1\rangle$ and $\langle B, R_2\rangle$ are called *isomorphic*, if there is a bijection $f : A \to B$ such that, for all $x, y \in A$,

$$xR_1y \iff f(x)R_2f(y).$$

¿From the point of view of the order properties, isomorphic orders are essentially the same.

Example 21 The order $\langle\{1, 2\}, \leq\rangle$ is isomorphic to $\langle\{2, 3\}, \leq\rangle$. The isomorphism is the function mapping 1 to 2 and 2 to 3.

Example 22 $\langle\mathbb{N}, \leq\rangle$ is not isomorphic to $\langle\mathbb{Z}, \leq\rangle$ and $\langle\{\frac{1}{n} : n \in \mathbb{N}\}, \leq\rangle$. However, $\langle\mathbb{N}, \leq\rangle$ is isomorphic to $\langle\{\frac{1}{n} : n \in \mathbb{N}\}, \geq\rangle$. What is the isomorphism function?

Excercise 25 In Exercise 24 draw the diagrams of non-isomorphic tree-like orders only.