بِسْمِ ٱللهِ ٱلْرَحْمٰنِ ٱلْرَحِيمِ

Treasures of the Islamic Scientific Tradition

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Books by Prof. ^{c}Alī A. al-Daffā^c

1. The Muslim Contribution to Mathematics,
   London 1978.
2. (with John J. Stroyls) Studies in the Exact Sciences in Islam,
   Dhahran and New York 1984.
3. (with J. Shawqi) العلوم الرياضية في الحضارة الإسلامية
   New York 1985.

These six lectures and two workshops:

- On specific topics in mathematics, and on astronomy.
- In mathematics, emphasis on original sources: manuscript texts, translation, pictures, instruments.
- We hope you take some of it home and tell about it to others.
- Attention to recent and current research, and what we do not yet know. Much has been lost …

Today's Lecture:

Conic sections and their applications in the medieval Islamic tradition.

Prejudice:

The Muslims only transmitted Greek mathematics to Western Europe

Facts:

1. Transmitting is NOT easy!
2. They did much more

Transmission of scientific knowledge into Arabic:

1. ca. 135/750 - 160/775: from Iran, Pahlavi
2. ca. 160/775 - 185/800: from India, Sanskrit
3. from 185/800 onwards: from Byzantium, Greek.

Banū Mūsā: three brothers, Muḥammad (died 259/873), Aḥmad, al-Ḥasan.
Their father, Mūsā ibn Shākir, had a high position at the court of Caliph al-Ma'mūn 197/813- 218/833.

The *Conics* of Apollonius of Perga (200 BC).

One of the most advanced works of theoretical Greek geometry (together with the works of Archimedes).

In 8 Books: Book 1-4 extant in Greek, Books 1-7 in Arabic

- Conic sections (hyperbola, parabola, ellipse) defined as plane sections of an oblique circular cone.
- Fundamental property is equivalent to modern equation of conic in oblique coordinate systems, with coordinate axes any tangent to the conic and the line joining the tangent to the center of the conic.
- Transformation of coordinates, tangency properties, asymptotes, radius of curvature, etc.

Report of their translation of the *Conics* of Apollonius. (around 235/850 - 255/870):

First, they had only one (very bad) Greek manuscript of Books 1-7, but failed to understand it.

al-Ḥasan developed for himself the theory of plane sections of a (circular) cylinder, and wrote a book about the theory, now lost. He proved:

1. for any plane section of a cylinder, there is a cone which intersects the same plane in the same section
2. for any closed plane seciton of a cone, there is a cylinder which intersects the same plane in the same section

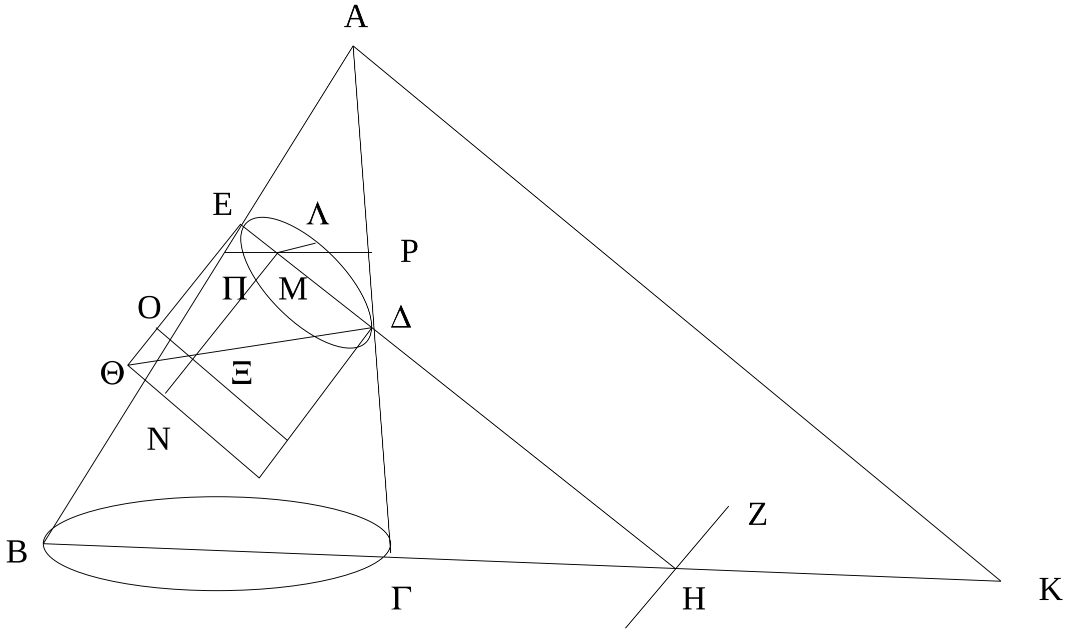
al-Ḥasan died.

Aḥmed went to Syria and found another Greek manuscript of Books 1-4 in the revised version of Eutocius of Ascalon (ca. 500 CE).

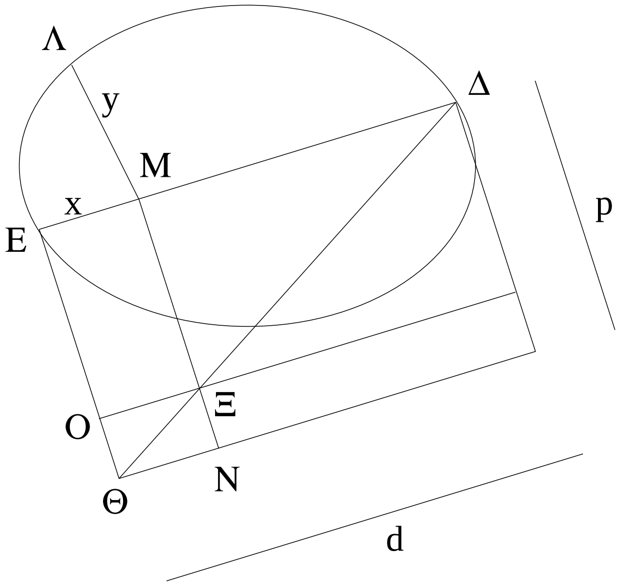
Aḥmad now succeeded in understanding Books 1-4, and then Books 5-7. He provided the (Greek) manuscript with references ("as was proved in Theorem x of Book y")

Aḥmad then had the work translated:
Booiks 1-4 by Hilāl ibn Abī Hilāl al-Ḥimṣī
Books 5-7 by the famous Thābit ibn Qurra al-Ḥarrānī (221/836 - 288/901)

An example: Proposition 13 of Book 1 of the *Conics*



$$\Lambda M^2 = \mathrm{ME}\mathrm{O}\Xi \qquad y^2 = px - (p/d)x^2$$



ellipse is the Greek word *elleipsis* (elleipein = “to be deficient”)
Arabic translation: قطع ناقص, “deficient section”.

$\iota\gamma'$. If a cone is cut by a plane through the axis, and by another plane meeting each side of the axial triangle, being neither parallel to the base nor subcontrary, and let the plane containing the base of the cone meet the cutting plane in a straight line perpendicular either to the base of the axial triangle or to its rectilinear extension; then any parallel line drawn from the section of the cone (parallel) to the common section of the planes as far as the diameter of the section will be equal in square to a certain area applied to a certain straight line, such that the diameter of the section will bear to it the same ratio as the square on the line drawn from the vertex of the cone parallel to the diameter of the section as far as the base of the triangle bears to the rectangle contained by the intercepts made by it on the sides of the triangle; the (area) having (as its) breadth the intercept made by it on the diameter in the direction of the vertex of the section, and being deficient (Greek: *elleipon*) by a figure similar and similarly situated to the rectangle contained by the diameter and the line along which they are equal in square; and let such a section be called an ellipse (Greek: *elleipsis*).

كل مخروط يعطع بسطحٍ يمرّ على سهمه ويعطع أيضًا بسطحٍ آخر يمرّ على كلي ضلعي المثلث المخرج على سهم المخروط ولا يكون موازيًا لقاعدة المخروط ولا مخالفًا لها ويكون السطح الذي فيه قاعدة المخروط والسطح الذي يقطع المخروط إذا أخرجا يلتقيان على خط يكون على زوايا قائمة إمّا على قاعدة المثلث المخرج على سهم المخروط وإما بالخط المتصل بالقاعدة على استقامة فإنّ الخط الذي يخرج من أي موضع كان من خط القطع إلى قطره إذا كان موازيًا للخط الذي يتقاطع عليه السطحان يقوى على سطحٍ مضافٍ إلى خطٍ يكون نسبة قطر القطع إليه كنسبة مربع الخط الذي يخرج من رأس المخروط موازيًا لقطر القطع ويلقى قاعدة المثلث الذي يمرّ على سهم المخروط إذا أُخرجت القاعدة على استقامة إلى الذي يكون من ضرب وتر الزاوية التي عند رأس المخروط ويحيط بها الخط الذي يخرج من رأس المخروط موازيًا لقطر القطع وأحد ضلعي المثلث الذي يمرّ على سهم المخروط في وتر الزاوية التي عند رأس المخروط أيضًا

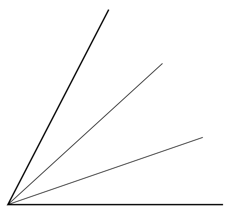
ويحيط بها الخط الذي خرج من رأس المخروط موازيًا لقطر القطع والضلع الآخر من المثلث الذي يقطع المخروط على سهمه ويكون عرض ذلك السطح الخط الذي يكون من موضع مسقط الخط المخرج من القطع على قطر القطع إلى رأس القطع ويكون ناقصًا سطحًا شبيهًا بالسطح الذي يكون من ضرب قطر القطع في الخط الذي يقوى عليه الخطوط المخرجة إلى قطر القطع على الترتيب وليُسَمّ هذا القطع الناقص

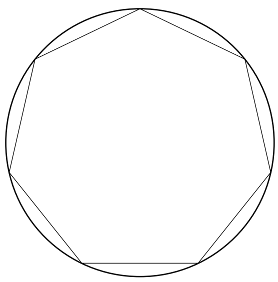
Applications of Conic Sections by Medieval Islamic Mathematicians.

New solutions to ancient Greek geometrical problems.

Trisection of the angle (new: related to $\sin 1^o$!): Al-Kūhī

Construction of the regular heptagon: Al-Kūhī, Ibn al-Haytham etc. (4th/10th-5th/11th c.)





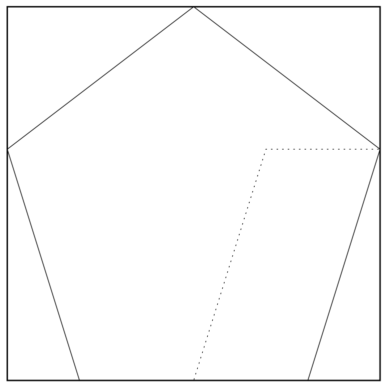
Trisection by Al-Kūhī is on the internet, in the webpage for these lectures:

www.math.uu.nl/people/hogend/treasures

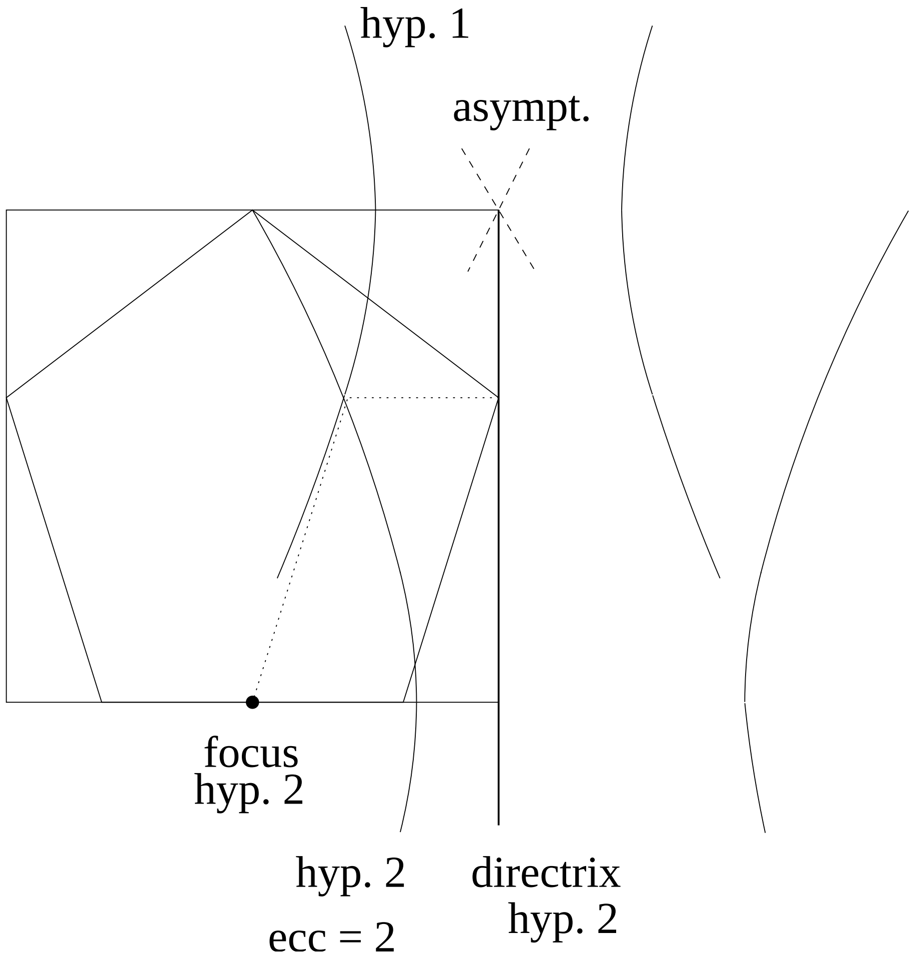
2. Solutions to new geometrical problems.

Example: Construction of an equilateral pentagon in a square

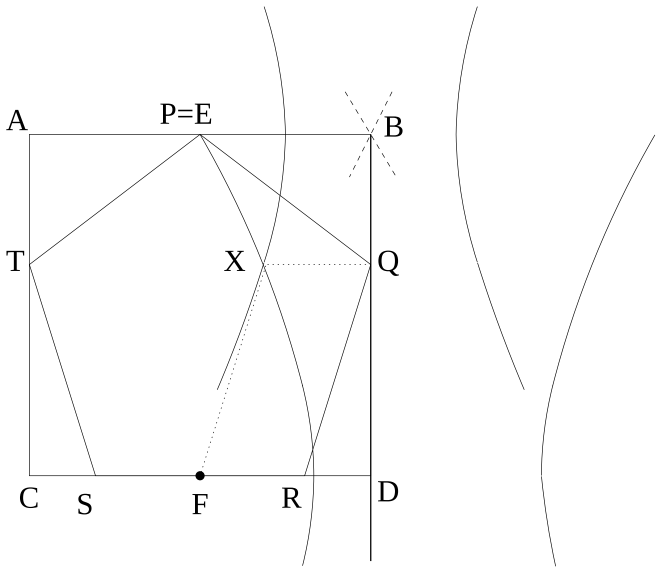
Abū Sahl al-Kūhī, Iran or Iraq, late 4th/10th c.



How did he do it?



Sketch of the construction of the equilateral pentagon $PQRST$ in a given square $ABCD$ by Abū Sahl al-Kūhī, ca. 375/985.



- Find the midpoints E and F of AB and CD.

- Analysis: Suppose we have found $PQRST$ such that $|PQ| = |QR| = |RS| = |ST| = |TP|$. By symmetry $P = E$ and $|RF| = |FS|$. Make the parallelogram $QRFX$. Then $|QX| = |FR| = \frac{1}{2}|QE|$. The collection of all points X such that $|QX| = \frac{1}{2}|QE|$ (where Q and X vary but QX is always perpendicular to BD) is a hyperbola with center B, axis on AB, and known asymptotes (one of which passes through F).

- We also have $|QX| = \frac{1}{2}|QR| = \frac{1}{2}|XF|$. The collection of all points X such that $|QX| = \frac{1}{2}|XF|$. is another hyperbola with focus F, directrix BD and eccentricity 2. (Problem: Apollonius did not discuss the focus-directrix property of the hyperbola, so al-Kūhī had to do that as well.)

- Construction: Draw these two hyperbolas. Choose the point X as the appropriate intersection. Then draw XF and drop perpendicular XQ. The rest is now easy. (Note that al-Kūhī and Apollonius drew only one branch of each hyperbola. The other branches have been drawn here for sake of clarity only.)

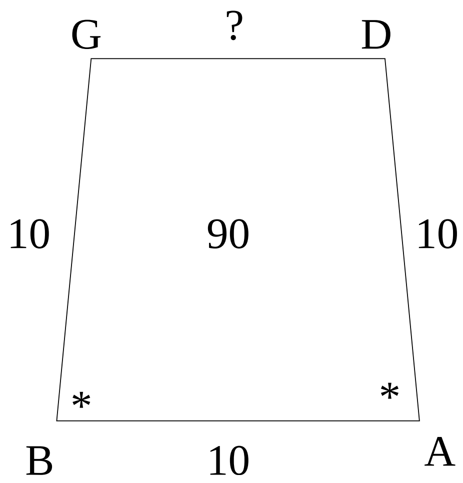
3. Algebra:

Solution of cubic equations (4th/10th c. and later), by ^{c}Umar al-Khayyām and others

Solution of some quartic equations; earliest in late 4th/10th century;
most of this work is lost.

We will now present a solution by an anonymous late-4th/10th century author. The unique Arabic manuscript is now in Holland. See the handout.

Problem in the anonymous Arabic text from the late 4th/10th century:

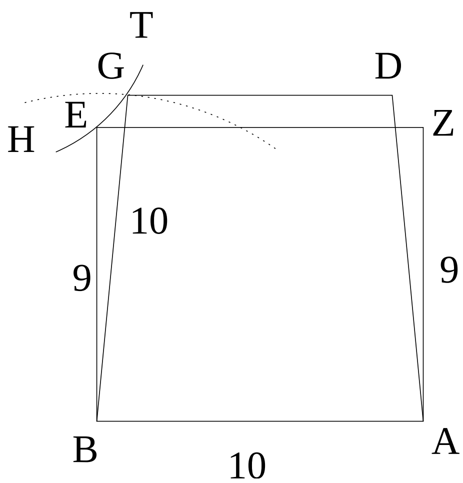


The anonymous author poses $? = 10 - 2x$ and finds

$$20x^3 + 1,900 = x^4 + 2,000x.$$

Solution of this problem by means of a hyperbola:

(See the analysis in [c]A. Al-Daffā,
العلوم الرياضية في الحضارة الاسلامية vol. 1, pp. 271-273:)



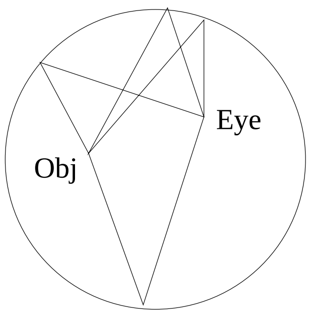
Put AB=10 and make a rectangle $ABEZ$ with sides 10 and 9.
Draw a circle with center B and radius 10, and a hyperbola through E with asymptotes AB and AZ. Let them intersect in G.
Draw GD parallel to BA and make $\angle GBA = \angle DAB$.

(Now you can find x !!)

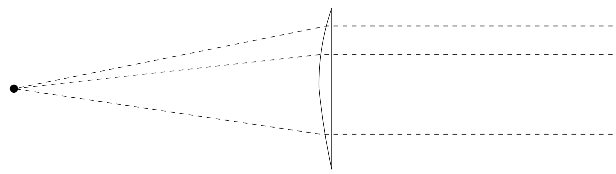
4. Optics:

Parabolic burning mirrors

Problem of Alhazen (= al-Ḥasan ibn al-Haytham, ca. 400/1000) to find the point of reflection in a given spherical mirror if the eye and the object are given.



Mysterious hyperbolic burning lenses (4th/10th c.; al-^{c}Alā' ibn Sahl)



A segment of a hyperbola, perpendicular to its axis, is rotated around its axis.
Make a lense of glass or crystal in the form of the resulting hyperbolic solid.

Assume that the hyperbola has the right eccentricity (corresponding to the physical properties of the glass or crystal).

Rays of sunlight parallel to the axis, will be refracted by the lense towards the focus of the hyperbola.
(The historical interpretation is problematic.)

5. Mosaic patterns:

A construction by Ibn al-Haytham (ca. 400/1010) has been preserved.

See the workshop: mysteries of Islamic mosaics.

6. Sundials:

7. Qibla instrument, using ellipses (programmed by Mr. Eelco Nederkoorn).

See the special lecture on this instrument, where you can receive a copy of this instrument designed by medieval Islamic mathematicians, and programmed by modern computers and sattelite images of the earth.

Now Mr. Nederkoorn will give a brief introduction.