**Workshop: Mysteries of Islamic mosaics.**

Introduction: There are very few medieval Islamic sources on mathematical problems of architecture and the geometrical construction of mosaics. The most important of these sources are

- A paper scroll, probably from Tabriz in Iran, and dating back to the 11th century Hijra, now in the Topkapı library in Istanbul.[1]
- A disorganized collection of forty pages of geometrical constructions with instructions in a Persian manuscript, also dating back to approximately the 11th century Hijra, now in the Bibliothèque Nationale in Paris (Persian manuscript *Ancien Fonds 169*).[2]

In this workshop we will discuss a few constructions from the second of these sources. We will see that there must have existed a sophisticated mathematical tradition among architects and medieval designers, about which almost nothing is known today.

Participants to this workshop need, in addition to this handout, a scissors, a protractor, a piece of paper, and preferably a ruler and glue as well. In order to save time during the workshop, two geometrical constructions using ruler and compasses have been performed in computer drawings. The participants who want to check these drawings are of course welcome to do so but will need a compass for this purpose.

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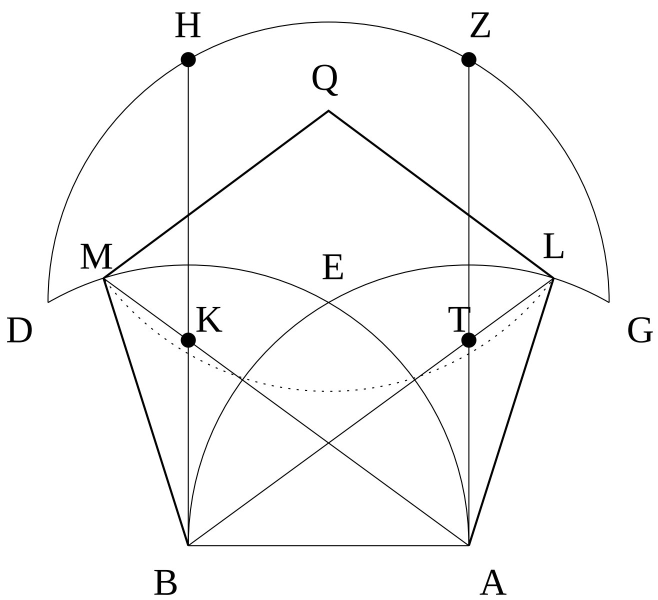
[1] It has been published in facsimile in: Gülru Necipoğlu, *The Topkapı Scroll, geometry and ornament in Islamic architecture: Topkapi Palace Museum Library MS. H. 1956.* Santa Monica, Ca. 90401-1455: Getty Center for the History of Art and the Humanities, 1995. ISBN 0-89236-335-5.

[2] The manuscript has been published in Persian in pp. 73-95, without commentary, by Seyyed Alireza Jazbi, *Applied Geometry: Abolvafa Mohammad ibn Mohammad Albuzjani, rewritten into modern Persian with appendices,* (هندسه ایرانی کاربُرد وهندسه در عمل) Tehran, Soroush Press, 1991. and translated into Russian in pp. 315-340 of M.S. Bulatov. *Geometricheskaya Garmonizatsiya v arkhitekture Srednei Azii IX - XV vv.* (Geometrical harmonisation in the architecture of Central Asia, 9th-15th centuries), Moskou: Nauka 1988. For partial analyses see Alpay Özdural, Omar Khayyam, Mathematicians and *Conversazioni* with Artisans, *Journal of the Society of Architectural Historians* **54** (1995), 54-71, and Alpay Özdural, On interlocking similar or corresponding figures and ornamental patterns of cubic equations, *Muqarnas* **13** (1996), 191-211.

*Page 186b from the Paris manuscript*

**Workshop: Mysteries of Islamic mosaics. Part 1.**




An easy construction of a regular pentagon with a fixed compass-opening, equal to the length of side $AB$. Source: the Paris manuscript, page 186b, see p. 2. Instructions in the manuscript (used in the computer drawing):

With centers $A$ and $B$, draw arcs $AED$ and $BEG$, to meet at $E$. With center $E$, draw arc $DHZG$. Find points $H$ and $Z$ on arc $DHZG$ such that $DH = GZ = AB$. Draw $BH$ and $AZ$. Find points $T$ and $K$ such that $ZT = HK = AB$ (Now $ZTKH$ is a square.)

Draw the straight line $BT$ and extend it to meet arc $BG$ at point $L$. Draw the straight line $AK$ and extend it to meet arc $AD$ at point $M$. Find $Q$ as the point of intersection of small sections of arcs with centers $L$ and $M$. (Then $QM = QL = AB$.) Draw straight lines $BM$, $AL$.

Then $ABMQL$ is a regular pentagon.

You can check the construction by drawing an arc with center $Q$; this arc must pass through the intersections of line $AM$ and arc $BG$, and of line $BL$ with arc $AD$. (This part was not understood by the scribe of the manuscript.)

*Question: Is the construction exact of approximate? Check the sides and angles! Can you prove that the angles are equal?*

*Mystery: How was the construction discovered?*

**Workshop: Mysteries of Islamic mosaics: Part 2.**

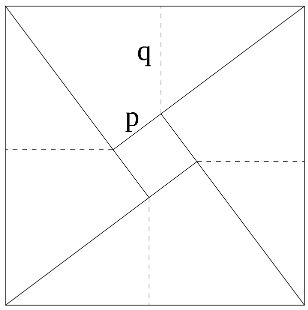
One page (f. 197a) of the Paris manuscript contains eight drawings without instructions. Each of the drawings shows a figure which is divided into a number of pieces, like a jigsaw-puzzle. The idea seems to be that these pieces can be put together in the form of another figure.

A copy has been attached of an eight-pointed star, which is one of the figures on the page. Because the instructions are missing, it is not possible to make a computer drawing, because it is not clear how exactly the figure should be drawn.

1. Cut out the parts of the eight-pointed star.
2. Try to put them together in the form of a regular octagon. (You may discover small errors in the figure during this process).
3. Do you think that the puzzle is mathematically exact or approximate? That is to say: is it possible to mathematically divide a (perfect) regular eight-pointed star as in the drawing, and put the pieces together in the form of a regular octagon?
4. *Question or mystery: If we have an eight-pointed star, how exactly do we make the drawing?* (the real problem is how to find the size of the square in the middle)?

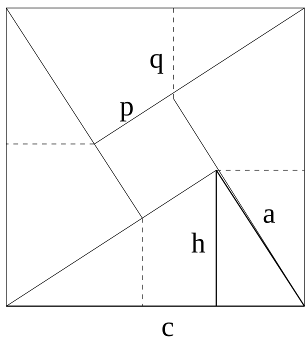
**Workshop: Mysteries of Islamic mosaics: Part 3.**

Introduction: On medieval Islamic buildings one often finds a square which is subdivided into four congruent right-angled triangles and a small square in the middle, as in the figure.




In general, the side of the small square $P$ will not be equal to its distance $Q$ between its anguar points and the sides of the big square.

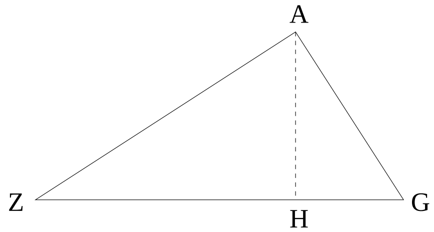
However, one can choose the angles and the size of the small square in such a way that $p = q$, as in the following figure. This produces an interesting pattern, which is discussed on many pages of the Persian manuscript.




If p=q then h+a=c

Exercise: Show that if $p = q$ as in the figure, we have $a + h = c$ in the right-angled triangle. (Hint: The answer is easy if you know how to find it!)

In a short treatise on algebra, $^c$Umar al-Khayyām (died 517/1123) constructed a right-angled triangle $AGZ$ (my notation) in such a way that angle $A$ is a right angle, and such that $|AH| + |AG| = |GZ|$, where $AH$ is the altitude from $A$. This is exactly the triangle in the mosaic pattern.

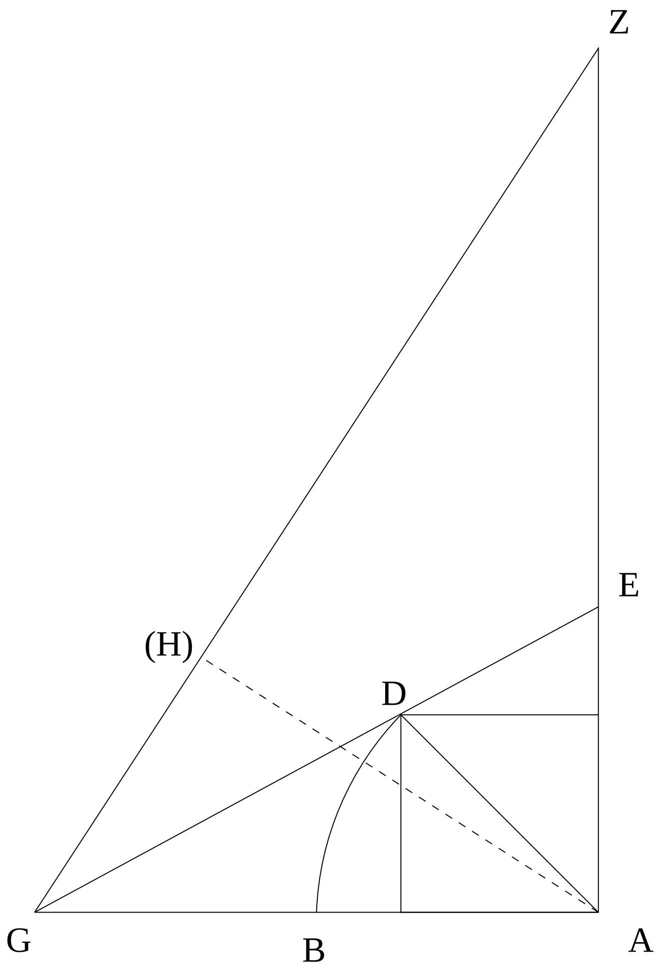



He supposed that he has found the triangle and he puts $GH = 10$ and $AH = x$. After some algebrical manipulations,[3] he deduces the cubic equation $x^3 + 200x = 20x^2 + 2000$. (In his words: 'a cube and two hundred things are equal to twenty capitals and two thousand of number")

He then solves this cubic equation geometrically by means of a hyperbola and a circle, see the appendix. The root $x$ cannot be constructed by means of ruler and compass. The Persian manuscript says on f. 191a that Ibn al-Haytham (died 432/1041) constructed the pattern with a parabola and a hyperbola, but no details are given. The following construction of triangle $AGZ$ is mentioned in the Persian manuscript: (letters in parentheses, such as $(H)$ do not occur in the manuscript)

[3]For the text see A. Djebbar, R. Rashed, *L'Oeuvre Algébrique d'al-Khayyām*, Aleppo 1981, pp. 85-87 (Arabic): By the theorem of Pythagoras $AG^2 = 10 + x^2$. Because angle $A$ is a right angle, we have $AG^2 = GZ \cdot GH$, so $GZ = 10 + x^2/10$. Since $GZ = GA + AH$ by assumption,

*Page 189b from the Paris manuscript*

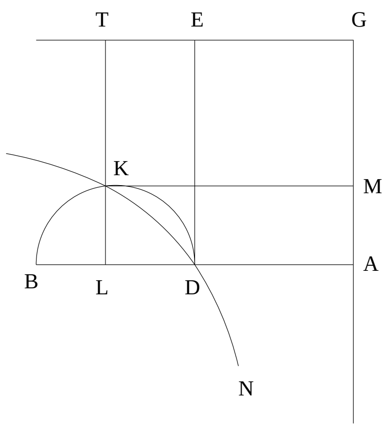



Draw a square $APDQ$. Draw the diagonal $AD$, and construct point $B$ on $AD$ extended in such a way that $AB = AD$. Then construct $G$ on $AB$ extended such that $BG = AB$. Draw $GD$ and extend it to meet the rectilinear extension of the other side of the square at point $E$. Choose point $Z$ on the rectilinear extension of $A$ in such a way that $EZ = AG$. Join $GZ$. These instructions have been used for the computer drawing.

The text then uses triangle $AGZ$ as a triangle with the required property $|AH| + |AG| = |GZ|$. Check whether this property holds in the figure. Have we achieved the impossible?

*Mystery 3: How was this construction discovered?*

**Appendix: Al-Khayyām's solution of** $x^3 + 200x = 20x^2 + 2000$




Al-Khayyām's solution is general, for $x^3 + bx = ax^2 + c$, but in the figure I have used $a = 20, b = 200, c = 2000$. Let $ADEG$ be a rectangle with sides $AD = c/b$ and $AG = \sqrt{b}$ and choose $B$ on $AD$ extended such that $AB = a$. Draw a semicircle with diameter $BD$ and draw a hyperbola through $D$ with asymptotes $AG, GE$. Let the semicircle and hyperbola intersect at $K$. Draw $TKL$ parallel to $GA$ to meet $GE$ and $AD$ at $T$ and $L$, and draw $KM$ parallel to $BA$ to meet $AG$ at $M$. Then $AL = x$. Sketch of the proof:

Because $D$, $K$ are on the hyperbola, the areas of rectangles $DEGA$ and $KTGM$ are equal (Apollonius, *Conics*, II:3); from this it can be shown that $AL : LT = DL : LK$. Since $K$ is on the circle, $LK^2 = DL \cdot LB$. Then $AL^2 : LT^2 = DL : LB$, or $x^2 : b = (x - c/b) : (a - x)$, hence $ax^2 - x^3 = bx - c$.

For the Arabic text, see A. Djebbar, R. Rashed, *L'Oeuvre Algébrique d'al-Khayyām*, Aleppo 1981, pp. 92-95 (Arabic).