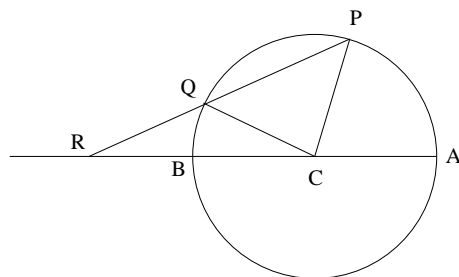


The trisection of the angle.

The trisection of the angle was one of the three famous problems of classical Greek geometry (the other two problems being the duplication of the cube and the quadrature of the circle). Around 300 B.C., Euclid showed in *Elements* I:10 how to divide a given angle into two equal parts by means of ruler and compass. The trisection of the angle is the problem to divide a given angle into three equal parts. The European mathematician Pierre L. Wantzel (1814-1848) proved in 1837, using the theory of algebraic equations, that the trisection of the angle is not in general possible by means of ruler and compasses (Kline p. 764). There are some special angles which can be trisected by ruler and compass, for example an angle of 90 degrees.

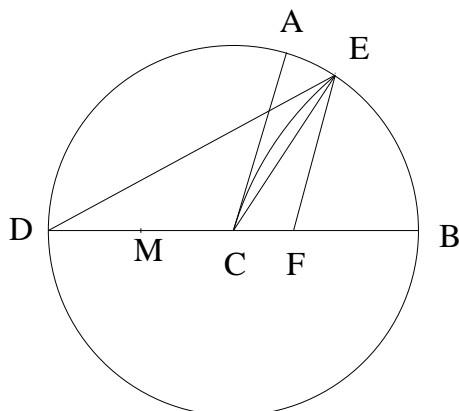
In the second or third century B.C. Greek geometers found trisections of the angle by other means (see Heath, vol. 1, pp. 235-244).

One Greek trisection can be reconstructed from the *Book of Lemmas* (kitāb al-ma'khūdhāt) which has been attributed to Archimedes and which has come down to us only in a medieval Arabic translation. (Figure 1) Draw a circle with centre C and let it be required to trisect the angle PCA with points P and A on the circle. Draw the diameter AB through A and extend it on the side of B . Now insert a segment QR equal to the radius of the circle between the outer side of the circle and the extended diameter, in such a way that point Q is on the circle between P and B and point R is on the rectilinear extension of AB . Then $\angle QCR$ is one-third of angle PCA . This construction is an example of a type of constructions called *neusis* in Greek geometry. A neusis is the insertion of a straight segment of given length between two given lines, in such a way that the segment or its rectilinear extension passes through a given point. Some Greek geometers (such as Archimedes) accepted this type of constructions without further justification.



Other Greek geometers preferred to work with conic sections. Two trisections of the angle by means of a circle and a hyperbola survive in the

Collections of Pappus of Alexandria (third century A.D.). One of these trisections was transmitted into Arabic, and it occurs in the works of Aḥmad ibn Mūsā and Thābit ibn Qurra (third century H.). A very simple trisection of the angle was invented by Abū Sahl al-Kūhī in the fourth century of the Hijra. This trisection was plagiarized by Aḥmad ibn Muḥammad ibn ʿAbdaljalīl al-Sijzī (for Arabic texts on the trisection see Knorr pp. 267-309 and for facsimiles of the manuscripts see Knorr pp. 358-363, 370, for al-Kūhī see also Sayılı).



Al-Kūhī supposes that $\angle ACB$ is the angle to be trisected (Figure 2). Draw a circle with C as centre and arbitrary radius and assume that points A and B are on this circle. Let BC extended meet the circle at D and let M be the midpoint of CD . He then draws the orthogonal hyperbola which has centre M , passes through C and has the property that line AC is tangent to it. This hyperbola will intersect the circle at a point E between A and B . Then $\angle EDB$ is one-third of angle ACB . Proof: Draw EF parallel to AC to meet BC at F , and draw EC .

Since point E is on the hyperbola we have (by Apollonius' *Conics* I:12) $EF^2 = FC \cdot FD$, so $FC : FE = FE : FD$. Because $\angle CFE = \angle EFD$, the triangles CFE and EFD are similar, so $\angle CEF = \angle EDF = \angle EDC$. Since point E is on the circle, we have $EC = CD$ so $\angle CED = \angle EDC$. Therefore $\angle ACB = \angle EFB = \angle FED + \angle EDF = 3\angle EDC$.

Al-Kūhī's assumption that the hyperbola can be drawn is based on the theory in the end of Book 1 of the *Conics* of Apollonius of Perga (200 B.C.) which had been translated into Arabic in the third century of the Hegira. Apollonius explains the three-dimensional construction of a cone which intersects the plane in the desired conic section. Al-Kūhī wrote a treatise on a

special type of compasses with which this type of construction could be carried out. There is no evidence that this so-called “perfect compass” (al-birkār al-tāmm) was really used in practice. In the time of al-Kūhī, the trisection of the angle was of theoretical importance only. This situation changed when it was realized that the trisection of the angle is related to the problem to compute the sine of one degree, a fundamental quantity in trigonometrical tables. If an angle of α degrees can be constructed by means of ruler and compass, one can always compute the sine of α with arbitrary accuracy by means of square root extractions. In this way one can compute e.g. the sine of 3 degrees.

Jamshīd Ghiyāth al-Dīn al-Kāshī showed in his treatise *On the Chord and the Sine* (fi’l-watar wa’l-jayb), which is now lost, but survives in later commentaries, that the trisection of a given angle can be reduced to the problem to solve a cubic equation with of the form $x^3 + q = px$ with $p, q > 0$ known quantities. He developed an iterative process to compute the root x of such an equation. In the case of the trisection of an angle of 3 degrees, this process produces a rapidly converging series of approximations to the sine of 1 degree. He then made the accurate value of the sine of 1 degree was the basis of a new sine table (Juschkeiwitsch pp. 319-323).

In Chapter 3 of Book 3 of al-Qānūn al-Mas‘ūdī, al-Bīrūnī studied the side of the regular nonagon in a circle with known radius, in order to compute the chord of 40 degrees, that is to say, two times the sine of 20 degrees (Al-Bīrūnī vol. 1, pp. 286-292). He showed that this quantity can also be found if either of the cubic equations $x^3 = 1 + 3x$ or $x^3 + 1 = 3x$ can be solved, and he gave an approximation of the root by means of which the sine of 20 degrees can be computed with an accuracy of 7 decimals (Schoy, pp. 78-82). This problem is equivalent to the trisection of an angle of 60 degrees (Juschkeiwitsch, p. 311)

In European mathematics, the trisection occurs again in the work of Francois Viète. He used it in order to solve the cubic equation of the form $x^3 + q = px$. The algebraic solution leads to complex numbers which Viète wanted to avoid (Kline, pp. 266-272).

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Duplication of the Cube

The duplication of the cube is one of the three famous problems of classical Greek geometry (the other two problems being the trisection of the angle and the quadrature of the circle). The duplication of the cube is to construct a cube whose volume is two times the volume of a given cube. This geometrical problem which made its first appearance in early Greek geometry before 450 B.C. or earlier. There are legends which suggest a religious origin of the problem: a certain altar had to be doubled in size in such a way that the shape is kept the same. Algebraically, the problem is equivalent to the construction of $\sqrt[3]{2}$, and it was first proved by Wantzel in 1837 that the problem cannot be solved by means of ruler and compass (Kline p. 764).

Hippocrates of Chios (ca. 450 B.C.) reduced the problem to the construction of two mean proportionals x and y between two given lines a and b , i.e. two segments such that $a : x = x : y = y : b$, and this is the problem which was solved by later Greek mathematicians. If $b = 2a$ we have $x^3 = 2a^3$ so x is the side of the cube whose volume is two times the volume of the cube with given side a .

In the fourth century BC and later, many constructions of two mean proportionals between two given lines were found by Greek geometers. Here is a list of the geometers and the means which they used in their construction (for details see Heath vol. 1, pp. 244-270):

- Archytas of Taras (first half of fourth century BC) gave a three-dimensional construction using the intersection of a cylinder, a right cone and a torus
- Menaechmus (ca. 350 BC) used conic sections, a hyperbola and a parabola. The idea of this solution is easy to describe algebraically. In modern terms. Menaechmus uses a point of intersection of the hyperbola with equation $xy = ab$ so $a : x = y : b$ and the parabola with equation $y^2 = bx$, to construct x and y such that $a : x = x : y = y : b$.
- A solution using an instrument which is basically a combination of rulers has been attributed to the philosopher Plato (427 - 347 B.C.). The attribution must be erroneous because Plato despised the use of such instruments.
- Another mechanical solution has been attributed to Eratosthenes (third century B.C.)

- A solution by means of two parabolas is to be found in a work by Diocles (first century B.C.) entitled *On Burning Mirrors*. The equations of the two parabolas are in modern coordinates $x^2 = ay$ and $y^2 = bx$.
- Diocles also provided another solution by means of a curve called the cissoid.
- Nicomedes (second century B.C.) gave a solution by means of a neusis (see for a discussion of the concept of neusis the article on the trisection of the angle)
- Another solution by means of a circle and a hyperbola is found in works of various geometers, including Apollonius of Perga (ca. 200 B.C.). This solution was well known to the medieval Islamic mathematicians.
- Another solution using a moving ruler is given by Sporus and Pappus (both third century A.D.)

Most of the Greek solutions were transmitted into Arabic through the commentary of Eutocius on Book 2 of Archimedes' *On the Sphere and Cylinder*, or otherwise, and some solutions were translated into Latin. Most medieval Islamic and Latin geometers contented themselves by studying the many existing solutions of the problem and they did not develop new solutions (except the Andalusian king al-Mu'taman ibn Hūd, who combined a solution by Menaechmus and a solution attributed to Apollonius, and obtained a construction of two mean proportionals by means of a parabola and a circle.). Later we find geometric solutions of the problem repeated in algebraic works, for example in the *Algebra* of Umar Khayyām in the discussion of the geometric solution of $x^3 = c$ (Daneshnameh-ye Khayyam pp. 209-210). For Umar Khayyām and his contemporaries and successors, the numerical approximation of $\sqrt[3]{c}$ was a more interesting problem than the geometric construction of the root x of $x^3 = c$, which was of theoretical importance only. Khayyām says that Ibn al-Haytham solved the problem to construct four mean proportionals $x_1 \dots x_4$ between two given lines a and b (Daneshnameh-ye Khayyam p. 236). Algebraically, this problem is equivalent to the solution of the equation $x_1^5 = a^4b$. In his *Algebra*, Khayyām also says that he described the numerical (approximative) solution of the equation $x^n = c$ in another work, which is now lost.

The 17th century European mathematicians were interested in generalizations of the problem. In his *Geometry* (1637), René Descartes studied the

problem to construct n mean proportionals between two given lines a and b , that is to say n segments $x_1 \dots x_n$ such that $a : x_1 = x_1 : x_2 = \dots = x_{n-1} : x_n = x_n : b$ (see Bos 1981 p. 309). Descartes studied solutions of this problem by means of algebraic curves and constructions of such curves. This tied in with the general subject to construct the roots of algebraic equations by means of curves. By 1750 A.D. this theory had died out (Bos 1984).

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The quadrature of the circle

The quadrature of the circle is one of the three classical problems of Greek geometry (the others being the duplication of the cube and the trisection of the angle). The quadrature of the circle is the problem to construct a square equal in area to a given circle. The problem can be solved if one can find a figure bounded by straight lines which is equal in area to a circle.

Generations of ancient Greek geometers struggled with this problem and variations on it. Around 450 BC, Hippocrates of Chios showed that it is possible to construct by means of ruler and compass, a square equal to a lunule of a certain type; a lune is a figure bounded by two arcs of a circle. He also showed that it is possible to find a square equal in area to a lune of another type plus a circle. Of course this does not lead to a solution of the quadrature of the circle (Heath vol. 1 pp. 183-200).

Antiphon of Athens (ca. 400 B.C.) observed that the quadrature of the circle may be solved approximately, because it is possible to construct squares equal in area to a regular polygons with 4,8,16 ... sides inscribed in a given circle. Deinostratos (middle of fourth century BC) used the quadratrix, a transcendental curve, for the quadrature of the circle. Archimedes (3d century B.C.) proved that the circle is equal in area to a right-angled triangle with basis equal to the circumference of the circle and height equal to the radius. Therefore the quadrature of the circle can be found if one can rectify the circle, i.e. to construct a straight line segment equal to the circumference of the circle. Archimedes constructed such a segment by means of a tangent to a spiral. None of the solutions thus far use the standard Euclidean means (ruler and compass); as a matter of fact, a solution by means of ruler and compass is impossible.

Archimedes also opened an approach to the problem which eventually turned out to be more fruitful. By considering inscribed and circumscribed regular 96-gons, he showed that the ratio between the circumference and the diameter of the circle (which will be indicated by the modern symbol π , introduced by W. Jones in 1706) can be estimated by $3\frac{10}{71} < \pi < 3\frac{1}{7}$. His approach was further developed by the mathematicians of medieval Islam. Jamshīd Ghiyāth al-Dīn al-Kāshī approximated π to 16 decimals by considering inscribed and circumscribed $3 \cdot 2^{28}$ -gons (Juschkewitch pp. 313-319). In practice, most people used the approximation $\pi \approx 3\frac{1}{7}$; Chinese mathematicians found $\pi \approx \frac{355}{113}$ which is accurate to 6 decimals. Medieval Islamic mathematicians began to wonder about the solubility of the problem. In

Book III of the Qānūn al-Masʿūdī, al-Bīrūnī conjectured that π is an irrational quantity, and Ibn al-Haytham suggested that although a square may exist equal to a given circle, it may not be possible for humans to find it.

Attempting to solve the age-old problem of the quadrature of the circle, European mathematicians in the 17th century found ways to represent π by series. Examples are the infinite product

$$\frac{4}{\pi} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \dots$$

found by John Wallis in his *Arithmetica Infinitorum* by interpolating in a triangle with binomial coefficients (in modern terms he studied values of the *Beta*-function at $(\frac{1}{2}, \frac{1}{2})$). Another example is the “arithmetical quadrature of the circle” by Gottfried Wilhelm Leibniz, with the result $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots$ (Katz pp. 525-527). This development is related to the new differential and integral calculus in the 17th century. Using the new calculus, similar but more complicated series were found and used to approximate π in a more efficient way than the method of Archimedes. For example, John Machin (1680-1752) found $\frac{\pi}{4} = \arctan \frac{1}{5} - \arctan \frac{1}{239}$ in 1706 (Beckman p. 145). This identity provides a rapidly converging series. The same developments were found (on a smaller scale and without the theoretical foundation of calculus) in Kerala (India) in the school of Mādhava from 1450 on (Katz pp. 494-496). In modern times, π has been approximated to several billions of decimals by means of computers (see for an introduction Berggren, Borwein and Borwein).

The problem of the “quadrature of the circle” was finally put to rest by Ferdinand Lindemann (1852-1939), who proved in 1882 that the number π is transcendental (Kline pp. 981-982). This means that it cannot be the root of an algebraic equation with integer coefficients. It follows that the geometric problem to find a square equal in area to a given circle can be solved neither by ruler and compass, nor by other algebraic curves (such as conic sections, which can be used to solve the trisection of the angle and the duplication of the cube.) Lindemann’s proof is quite complicated but later simpler proofs have been found by Niven, which can be understood by any graduate student in mathematics. (Niven’s proof that π is an irrational number uses simple calculus and can be understood by any beginning student in mathematics and physics; see Berggren, Borwein and Borwein p. 276.)

Janos Bolyai (1802-1860) showed in paragraph 41 of his *Appendix* (that is his treatise on non-Euclidean geometry which appeared in 1832) that the

“quadrature of the circle” is possible for some circles in Non-Euclidean geometry, because the area of such circles is equal to $c \cdot r^2$ with c a variable number depending on the radius (Stäckel pp. 214-216).

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