On the Failure of Fixed-Point Theorems for Chain-complete Lattices in the Effective Topos

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Abstract

In the effective topos there exists a chain-complete distributive lattice with a monotone and progressive endomap which does not have a fixed point. Consequently, the Bourbaki-Witt theorem and Tarski's fixed-point theorem for chain-complete lattices do not have constructive (topos-valid) proofs.

1 Introduction

In this note I show that in the effective topos Eff [2] there is a chain-complete distributive lattice with a monotone and progressive endomap which does *not* have a fixed point. An immediate consequence of this is that several fixed-point theorems for chain-complete posets have no constructive (topos-valid) proofs, cf. Section 5.

The outline of the argument is as follows. In Eff every chain is a discrete object in the sense of [3], hence it has at most countably many global points. Consequently, the poset $\nabla\omega_1$ is chain-complete in the effective topos, even though it is only countably complete in Set. The successor function on $\nabla\omega_1$ is monotone and progressive, and obviously does not have a fixed point.

We work out the details of the above argument carefully in order not to confuse external and internal notions of chain-completeness, discreteness, and countability. For the uninitiated, we have included a brief overview of the effective topos in Appendix A.

2 Preliminary observations

Let $2 = \{0, 1\}$ be the set with two elements. An object $X = (|X|, =_X)$ in Eff is *orthogonal to* $\nabla 2$ when the diagonal map $X \to X^{\nabla 2}$ is an isomorphism.[1] In the

[1] Such objects are also called *discrete*, see [3].

internal language of Eff the condition may be expressed by the formula

$$\forall f \in X^{\nabla 2} . \forall p \in \nabla 2 . f(p) = f(1).$$

The object $X^{\nabla 2}$ is described explicitly as the set $|X|^2$ with the equality predicate

$$[(x_0, y_0) =_{X^{\nabla 2}} (x_1, y_1)] = [x_0 =_X x_1] \cap [y_0 =_X y_1].$$

Let us compute exactly how universal quantification over $X^{\nabla 2}$ and $\nabla 2$ works. If $\phi : 2 \times |X| \to \mathcal{P}(\mathbb{N})$ is a strict extensional relation on $\nabla 2 \times X$ then $\forall p \in \nabla 2 . \phi(p, x)$ is represented by the strict extensional relation

$$x \mapsto \phi(0, x) \cap \phi(1, x).$$

If $\phi : |X|^2 \times |Y| \to \mathcal{P}(\mathbb{N})$ is a strict extensional relation on the object $X^{\nabla 2} \times Y$ then $\forall f \in X^{\nabla 2} . \phi(f, y)$ is represented by the strict extensional relation on Y which maps $y \in |Y|$ to

$$\textstyle\bigcap_{x_0, x_1 \in |X|} ([x_0 =_X x_0] \wedge [x_1 =_X x_1] \Rightarrow \phi(x_0, y) \cap \phi(x_1, y)).$$

The object $B = (\{0, 1\}, =_B)$ with

$$[x =_B y] = \begin{cases} \{0\} & \text{if } x = y = 0, \\ \{1\} & \text{if } x = y = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

is isomorphic to $1+1$. We call it the object of *Boolean values*. By the *uniformity principle* [5, 3.2.21], the following statement is valid in the internal logic of Eff: for all $\phi \in \mathsf{P}(\nabla 2 \times B)$, if $\forall p \in \nabla 2 . \exists d \in B . \phi(p, d)$ then $\exists d \in B . \forall p \in \nabla 2 . \phi(p, d)$.

Lemma 1 *The following statement is valid in the internal logic of* Eff*: for all* $\phi, \psi : \nabla 2 \to \Omega$*, if* $\forall p \in \nabla 2 . (\phi(p) \vee \psi(p))$ *then* $\forall p \in \nabla 2 . \phi(p)$ *or* $\forall p \in \nabla 2 . \psi(p)$.

Proof. We argue internally in Eff. Suppose $\forall p \in \nabla 2 . (\phi(p) \vee \psi(p))$ Then

$$\forall p \in \nabla 2 . \exists d \in 2 . ((d = 0 \wedge \phi(p)) \vee (d = 1 \wedge \psi(p))).$$

By the uniformity principle

$$\exists d \in 2 . \forall p \in \nabla 2 . ((d = 0 \wedge \phi(p)) \vee (d = 1 \wedge \psi(p))).$$

Consider such a $d \in 2$. If $d = 0$ then $\forall p \in \nabla 2 . \phi(p)$, and if $d = 1$ then $\forall p \in \nabla 2 . \psi(p)$. $\square$

For an object X and variable D ranging over $\mathsf{P}(X)$, let $\mathsf{orth}_{\nabla 2}(D)$ be the following formula in the internal language of Eff:

$$\forall f \in X^{\nabla 2} . (\forall p \in \nabla 2 . f(p) \in D) \implies (\forall p \in \nabla 2 . f(p) = f(1)).$$

We compute a strict extensional relation O which represents $\mathsf{orth}_{\nabla 2}(-)$ in the case $X = \nabla S$. The underlying set of $\mathsf{P}(\nabla S)$ is $\mathcal{P}(\mathbb{N})^S$, and every $D : S \to \mathcal{P}(\mathbb{N})$ is strict and extensional with respect to ∇S. Thus our strict extensional relation O takes $D : S \to \mathcal{P}(\mathbb{N})$ to

$$O(D) = \bigcap_{x_0, x_1 \in S} D(x_0) \cap D(x_1) \Rightarrow \{n \in \mathbb{N} \mid x_0 = x_1\}.$$

This is an inhabited set if, and only if, $x_0 \neq x_1$ implies $D(x_0) \cap D(x_1) = \emptyset$ for all $x_0, x_1 \in S$. Consequently, if $O(D) \neq \emptyset$ then there are at most countably many $x \in S$ for which $D(x) \neq \emptyset$.

In the internal language, define the object of subobjects of X orthogonal to $\nabla 2$ as

$$\mathsf{Orth}_{\nabla 2}(X) = \{D \in \mathsf{P}(X) \mid \mathsf{orth}_{\nabla 2}(D)\}.$$

When $X = \nabla S$, the object $\mathsf{Orth}_{\nabla 2}(\nabla S)$ has the underlying set $\mathcal{P}(\mathbb{N})^S$ and the equality predicate

$$[D =_{\mathsf{Orth}_{\nabla 2}(\nabla S)} E] = (D \Rightarrow E) \wedge (E \Rightarrow D) \wedge O(D).$$

For a set S let $\mathcal{P}_\omega(S)$ be the family of countable subsets of S.

Lemma 2 *Suppose S is a set and let $\mathsf{cl}_{\neg\neg} : \mathsf{P}(\nabla S) \to \nabla\mathcal{P}(S)$ be the $\neg\neg$-closure operator. The restriction of $\mathsf{cl}_{\neg\neg}$ to $\mathsf{Orth}_{\nabla 2}(\nabla S)$ factors through $\nabla\mathcal{P}_\omega(S)$:*

$$\begin{tikzcd}
\mathsf{Orth}_{\nabla 2}(\nabla S) \arrow[r, "i"] \arrow[d, dashed] & \mathsf{P}(\nabla S) \arrow[d, "\mathsf{cl}_{\neg\neg}"] \\
\nabla\mathcal{P}_\omega(S) \arrow[r, "\nabla j"] & \nabla\mathcal{P}(S)
\end{tikzcd}$$

Proof. In the diagram above j is the inclusion $\mathcal{P}_\omega(S) \subseteq \mathcal{P}(S)$. Recall that $\neg\neg$ as a morphism $\Omega \to \nabla 2$ is represented by the functional relation $F : \mathcal{P}(\mathbb{N}) \times 2 \to \mathcal{P}(\mathbb{N})$ defined by $F(P, q) = [f(p) =_{\nabla 2} q]$, where

$$f(p) = \begin{cases} 1 & \text{if } p \neq \emptyset, \\ 0 & \text{if } p = \emptyset. \end{cases}$$

The operator $\mathsf{cl}_{\neg\neg} : \mathsf{P}(\nabla S) \to \nabla\mathcal{P}(S)$ is composition with $\neg\neg$. It is represented by the functional relation $G : \mathcal{P}(\mathbb{N})^S \times \mathcal{P}(S) \to \mathcal{P}(\mathbb{N})$, defined by $G(P, Q) = [g(P) =_{\nabla\mathcal{P}(S)} Q]$ where

$$g(P) = \{x \in S \mid P(x) \neq \emptyset\}.$$

Notice that, for all $P_1, P_2 : S \to \mathcal{P}(\mathbb{N})$, if

$$\models (P_1 \Rightarrow P_2) \wedge (P_2 \Rightarrow P_1)$$

then $g(P_1) = g(P_2)$ (this is just extensionality of G).

The inclusion $i : \mathsf{Orth}_{\nabla 2}(\nabla S) \to \mathsf{P}(\nabla S)$ is represented by the functional relation $I : \mathcal{P}(\mathbb{N})^S \times \mathcal{P}(\mathbb{N}^S) \to \mathcal{P}(\mathbb{N})$, defined by $I(D, E) = [D =_{\mathsf{Orth}_{\nabla 2}(\nabla S)} E]$. The composition $\mathsf{cl}_{\neg\neg} \circ i$ is represented by the functional relation $K : \mathcal{P}(\mathbb{N})^S \times \mathcal{P}(S) \to \mathcal{P}(\mathbb{N})$ defined by

$$K(D, Q) = O(D) \wedge [g(D) =_{\nabla \mathcal{P}(S)} Q].$$

Now define $H : \mathcal{P}(\mathbb{N})^S \times \mathcal{P}_\omega(S) \to \mathcal{P}(\mathbb{N})$ by

$$H(D, Q) = O(D) \wedge [g(D) =_{\nabla \mathcal{P}(S)} Q].$$

Recall that $O(D) \neq \emptyset$ implies that there are at most countably many $x \in S$ for which $D(x) \neq \emptyset$. This implies that H is a total relation. It is in fact a functional relation representing a morphism $h : \mathsf{Orth}_{\nabla 2}(\nabla S) \to \nabla \mathcal{P}_\omega(S)$. It is easy to verify that h is the required factorization of $\mathsf{cl}_{\neg\neg} \circ i$ through ∇j. $\square$

3 Posets and Chains in the Effective Topos

In this section we work in the internal logic of the effective topos. First we recall several standard order-theoretic notions. A *poset* $(L, \leq)$ is an object L with a relation $\leq$ which is reflexive, transitive, and antisymmetric. A *lattice* $(L, \leq, \wedge, \vee)$ is a poset in which every elements $x, y \in L$ have a greatest lower bound $x \wedge y$, and least upper bound $x \vee y$. Note that a lattice need not have the smallest and the greatest element. A lattice is *distributive* if $\wedge$ and $\vee$ satisfy the distributivity laws $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. An endomap $f : L \to L$ on a poset $(L, \leq)$ is *monotone* when

$$\forall\, x, y \in L\,.\,(x \leq y \implies f(x) \leq f(y))\,,$$

and *progressive* when $\forall\, x \in L\,.\, x \leq f(x)$.

For $x \in L$ and $S \in \mathsf{P}(L)$ define $\mathsf{bound}(x, S)$ to be the relation

$$\mathsf{bound}(x, S) \iff \forall\, y \in L\,.\,(y \in S \implies y \leq x)\,.$$

We say that $z \in L$ is the *supremum* of $S \in \mathsf{P}(L)$ when

$$\mathsf{bound}(z, S) \wedge \forall\, y \in L\,.\,(\mathsf{bound}(y, S) \implies y \leq z)\,.$$

Lemma 3 *Suppose $(L, \leq)$ is a poset with a $\neg\neg$-stable order. For all $S \in \mathsf{P}(L)$ and $x \in L$, if x is the supremum of $\mathsf{cl}_{\neg\neg} S$ then x is the supremum of S.*

Proof. If $\leq$ is $\neg\neg$-stable then

$$\begin{aligned}
\mathsf{bound}(x, \mathsf{cl}_{\neg\neg} S) &\iff \forall\, y \in L\,.\,(\neg\neg(y \in S) \implies y \leq x) \\
&\iff \forall\, y \in L\,.\,(y \in S \implies \neg\neg(y \leq x)) \\
&\iff \forall\, y \in L\,.\,(y \in S \implies y \leq x) \\
&\iff \mathsf{bound}(x, S).
\end{aligned}$$

Because $\mathsf{cl}_{\neg\neg}S$ and S have the same upper bounds, if x is the supremum of one of them then it is the supremum of the other as well. $\square$

By a *chain* in a poset $(L, \leq)$ we mean $C \in \mathsf{P}(L)$ such that

$$\forall\, x, y \in L\,.\,(x \in C \wedge y \in C \implies x \leq y \vee y \leq x)\,.$$

The *object of chains in* L is defined as

$$\mathsf{Ch}(L) = \{C \in \mathcal{P}(P) \mid \forall\, x, y \in L\,.\,(x \in C \wedge y \in C \implies x \leq y \vee y \leq x)\}.$$

Proposition 4 *Every chain is orthogonal to* $\nabla\mathsf{2}$*, i.e.,* $\mathsf{Ch}(L) \subseteq \mathsf{Orth}_{\nabla\mathsf{2}}(L)$.

Proof. Consider any $C \in \mathsf{Ch}(L)$ and $f : \nabla\mathsf{2} \to L$ such that $\forall\, p \in \nabla\mathsf{2}\,.\, f(p) \in C$. We need to show that f is constant. Because C is a chain we have

$$\forall\, p, q \in \nabla\mathsf{2}\,.\,(f(p) \leq f(q) \vee f(q) \leq f(p))\,.$$

By a double application of Lemma 1 we obtain

$$(\forall\, p, q \in \nabla\mathsf{2}\,.\, f(p) \leq f(q)) \vee (\forall\, p, q \in \nabla\mathsf{2}\,.\, f(q) \leq f(p)).$$

Because $\leq$ is antisymmetric, either of these two cases implies $f(p) = f(q)$ for all $p, q \in \nabla\mathsf{2}$, as required. $\square$

4 The poset $\nabla\omega_1$

Let $(\omega_1, \preceq)$ be the distributive lattice of countable ordinals in Set. This is not a chain-complete poset, but it is complete with respect to countable subsets. More precisely, if $\mathcal{P}_\omega(\omega_1)$ is the family of all countable subsets of ω_1 then there is a map $\sup : \mathcal{P}_\omega(\omega_1) \to \omega_1$ such that $\sup(S)$ is the supremum of $S \in \mathcal{P}_\omega(\omega_1)$.

The object $\nabla\omega_1$, ordered by $\nabla\preceq$, is a distributive lattice in Eff. One way to see this is to observe that ∇ preserves finite products, therefore it maps models of the equational theory of distributive lattices to models of the same theory. Moreover, observe that ∇ preserves the negative fragment of logic ($\forall$, $\wedge$, $\implies$) and that statement "x is the supremum of S" may be written in that fragment. Therefore, the statement

$$\forall\, S \in \nabla\mathcal{P}_\omega(\omega_1)\,.\,\text{“}\nabla\sup(S) \text{ is the supremum of } S\text{”}$$

is valid in the internal language of Eff.

Lemma 5 *The poset* $\nabla\omega_1$ *is chain-complete in* Eff.

Proof. We claim that the supremum operator $\mathsf{Ch}(\nabla\omega_1) \to \nabla\omega_1$ is the composition

$$\mathsf{Ch}(\nabla\omega_1) \xrightarrow{\subseteq} \mathsf{Orth}_{\nabla 2}(\nabla\omega_1) \xrightarrow{\mathsf{cl}_{\neg\neg}} \nabla(\mathcal{P}_\omega(\omega_1)) \xrightarrow{\nabla\sup} \nabla\omega_1$$

The arrows marked by $\subseteq$ and $\mathsf{cl}_{\neg\neg}$ come from Lemmas 4 and 2, respectively.

We argue in the internal language of Eff. Consider any $C \in \mathsf{Ch}(\nabla\omega_1)$. Then $\mathsf{cl}_{\neg\neg}C \in \mathcal{P}_\omega(\omega_1)$, therefore $x = (\nabla\sup)(\mathsf{cl}_{\neg\neg}C)$ is the supremum of $\mathsf{cl}_{\neg\neg}C$. But since the order on $\nabla\omega_1$ is $\neg\neg$-stable x is also the supremum of C by Lemma 3. □

Corollary 6 *In the effective topos, there is a chain-complete poset with a monotone and progressive endomap which does* not *have a fixed point.*

Proof. Consider $\nabla\omega_1$ and the successor map. □

5 Consequences

The following theorems *cannot* be proved constructively, i.e., in higher-order intuitionistic logic with Dependent Choice:

1. Knaster-Tarski Theorem [4] for chain-complete lattices: a monotone map on a chain-complete lattice has a fixed point.
2. Bourbaki-Witt theorem [1, 6]: a progressive map on a chain-complete poset has a fixed point above every point.

References

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A The Effective Topos

We rely on [5] as a reference on the effective topos and give only a quick overview of the basic constructions here.

A.1 Definition of the effective topos

Recall that a *non-standard* predicate on a set X is a map $P : X \to \mathcal{P}(\mathbb{N})$, where we think of $P(x)$ as the set of realizers (Gödel codes of programs) which witness the fact that x has the property P. The non-standard predicates on X form a Heyting prealgebra $\mathcal{P}(\mathbb{N})^X$ with the partial order

$$P \leq Q \iff \exists\, n \in \mathbb{N} \,.\, \forall\, x \in X \,.\, \forall\, m \in P(x) \,.\, \varphi_n(m){\downarrow} \land \varphi_n(m) \in Q(x),$$

where φ_n is the n-th partial recursive function and $\varphi_n(m){\downarrow}$ means that $\varphi_n(m)$ is defined. In words, P entails Q if there is a program that translates realizers for $P(x)$ to realizers for $Q(x)$, uniformly in x. Predicates P and Q are *equivalent*, written $P \equiv Q$, when $P \leq Q$ and $Q \leq P$. If we quotient $\mathcal{P}(\mathbb{N})^X$ by $\equiv$ we obtain an honest Heyting algebra, but we do not do that.

Let $\langle -, - \rangle$ be a computable pairing function on the natural numbers $\mathbb{N}$, e.g., $\langle m, n \rangle = 2^m(2n + 1)$. The Heyting prealgebra structure of $\mathcal{P}(\mathbb{N})^X$ is as follows:

$$\begin{aligned} \top(x) &= \mathbb{N} \\ \bot(x) &= \emptyset \\ (P \land Q)(x) &= \{\langle m, n \rangle \mid m \in P(x) \land n \in Q(x)\} \\ (P \lor Q)(x) &= \{\langle 0, n \rangle \mid n \in P(x)\} \cup \{\langle 1, n \rangle \mid n \in Q(x)\} \\ (P \Rightarrow Q)(x) &= \{n \in \mathbb{N} \mid \forall\, m \in P(x) \,.\, \varphi_n(m){\downarrow} \land \varphi_n(m) \in Q(x)\}. \end{aligned} \tag{1}$$

We say that a non-standard predicate P is *valid* if $\top \leq P$, in which case we write $\models P$. The condition $\top \leq P$ is equivalent to requiring that $\bigcap_{x \in X} P(x)$ contains at least one number. Often a non-standard predicate is given as a map $x \mapsto \phi(x)$ where ϕ is an expression with a free variable x. In this case we abuse notation and write $\models \phi(x)$ instead of $\models \lambda x : X \,.\, \phi(x)$. In other words, free variables are to be implicitly abstracted over.

An object $X = (|X|, =_X)$ in the effective topos is a set $|X|$ with a non-standard *equality predicate* $=_X \,:\, |X| \times |X| \to \mathcal{P}(\mathbb{N})$, which is required to be symmetric and transitive (where we write $[x =_X y]$ instead of $x =_X y$ for better readability):

$$\models [x =_X y] \Rightarrow [y =_X x], \tag{symmetric}$$

$$\models [x =_X y] \land [y =_X z] \Rightarrow [x =_X z]. \tag{transitive}$$

Usually we write $\mathsf{E}_X(x)$ for $[x =_X x]$. Think of E_X as an "existence predicate", and $\mathsf{E}_X(x)$ as the set of realizers which witness the fact that x exists.

In the effective topos a morphism $F : X \to Y$ is represented by a non-standard functional relation $F : X \times Y \to \mathcal{P}(\mathbb{N})$. More precisely, we require that

$$\begin{aligned} &\models F(x,y) \Rightarrow \mathsf{E}_X(x) \wedge \mathsf{E}_Y(y) && \text{(strict)} \\ &\models [x =_X x'] \wedge F(x,y) \wedge [y =_Y y'] \Rightarrow F(x',y') && \text{(extensional)} \\ &\models F(x,y) \wedge F(x,y') \Rightarrow [y =_X y'] && \text{(single-valued)} \\ &\models \mathsf{E}_X(x) \Rightarrow \textstyle\bigcup_{y \in Y} \mathsf{E}_Y(y) \wedge F(x,y). && \text{(total)} \end{aligned}$$

Two such functional relations F, F' represent the same morphism when $F \leq F'$ and $F' \leq F$ in the Heyting prealgebra $\mathcal{P}(\mathbb{N})^{X \times Y}$. Composition of $F : X \to Y$ and $G : Y \to Z$ is the functional relation $G \circ F$ given by

$$(G \circ F)(x,z) = \textstyle\bigcup_{y \in Y} F(x,y) \wedge G(y,z).$$

The identity morphism $I : X \to X$ is the relation $I(x,y) = [x =_X y]$.

A.2 Interpretation of first-order logic in Eff

The effective topos supports an interpretation of intuitionistic first-order logic, which we outline in this section.

Each subobject of an object X is represented by a *strict extensional predicate*, which is a non-standard predicate $P : X \to \mathcal{P}(\mathbb{N})$ that satisfies:

$$\begin{aligned} &\models P(x) \Rightarrow \mathsf{E}_X(x), && \text{(strict)} \\ &\models P(x) \wedge [x =_X x'] \Rightarrow P(x'). && \text{(extensional)} \end{aligned}$$

Such a predicate represents the subobject determined by the mono $I : Y \to X$ where $|Y| = |X|$, $[x =_Y y] = [x =_X y] \wedge P(x)$, and $I(x,y) = P(x) \wedge [x =_X y]$. Strict predicates represent the same subobject precisely when they are equivalent as elements of the Heyting prealgebra $\mathcal{P}(\mathbb{N})^X$.

The interpretation of first-order logic with equality in Eff may be expressed in terms of strict extensional predicates and non-standard equality predicates. Suppose ϕ is a formula with a free variable x ranging over an object X.[2] The interpretation of ϕ is the subobject of X represented by the non-standard predicate $[\![\phi]\!] : |X| \to \mathcal{P}(\mathbb{N})$, defined inductively on the structure of ϕ as follows. The propositional part in the topos is interpreted by the Heyting prealgebra

[2] In the general case ϕ may contain free variables $x_1, \ldots, x_n$ ranging over objects $X_1, \ldots, X_n$, respectively. In this case ϕ is interpreted as a subobject of $X_1 \times X_n$. It is easy to work out the details once you have seen the case of a single variable.

structure of non-standard predicates, cf. (1):

$$\begin{aligned}
[\![\top]\!] &= \top \\
[\![\bot]\!] &= \bot \\
[\![\theta \wedge \psi]\!] &= [\![\theta]\!] \wedge [\![\psi]\!] \\
[\![\theta \vee \psi]\!] &= [\![\theta]\!] \vee [\![\psi]\!] \\
[\![\theta \Rightarrow \psi]\!] &= [\![\theta]\!] \Rightarrow [\![\psi]\!].
\end{aligned}$$

Suppose ψ is a formula with free variables x of type X and y of type Y, and let $P = [\![\psi]\!] : |X| \times |Y| \to \mathcal{P}(\mathbb{N})$ be its interpretation. Then the interpretation of the quantifiers is:

$$\begin{aligned}
[\![\exists\, x \in X \,.\, \psi]\!](v) &= \textstyle\bigcup_{u \in |X|} \mathsf{E}_X(u) \wedge P(u, v), \\
[\![\forall\, x \in X \,.\, \psi]\!](v) &= \textstyle\bigcap_{u \in |X|} \mathsf{E}_X(u) \Rightarrow P(u, v).
\end{aligned}$$

Suppose $f, g : X \to Y$ are morphisms represented by functional relations $F, G : X \times Y \to \mathcal{P}(\mathbb{N})$. The atomic formula $f(x) = g(x)$, where x is a variable of type X, is interpreted as the subobject of X represented by the non-standard predicate $[\![f(x) = g(x)]\!] : |X| \to \mathcal{P}(\mathbb{N})$, defined by

$$[\![f(x) = g(x)]\!](u) = \textstyle\bigcup_{v \in |Y|} (F(u, v) \wedge G(u, v)).$$

If other atomic predicates appear in a formula, their interpretation must be given in terms of corresponding strict extensional predicates.

This concludes the interpretation of first-order logic. The interpretation is sound for intuitionistic reasoning.

Lastly, let us give a description of powerobjects in the effective topos. If X is an object then the *powerobject* $\mathsf{P}(X)$ is the set $\mathcal{P}(\mathbb{N})^{|X|}$ with non-standard equality predicate

$$\begin{aligned}
[P =_{\mathsf{P}(X)} Q] = {} & (P \Rightarrow Q) \wedge (Q \Rightarrow P) \wedge \\
& \left(\textstyle\bigcap_{x \in |X|} P(x) \Rightarrow \mathsf{E}_X(x)\right) \wedge \left(\textstyle\bigcap_{x, y \in |X|} P(x) \wedge [x =_X y] \Rightarrow P(y)\right).
\end{aligned}$$

The complicated part in the second line says that P is strict and extensional. If x and y are variables of type X and $\mathsf{P}(X)$, respectively, then the atomic predicate $x \in y$ is represented by the strict extensional predicate $E : |X| \times \mathcal{P}(\mathbb{N})^{|X|} \to \mathcal{P}(\mathbb{N})$ defined by $E(u, P) = \mathsf{E}_X(u) \wedge \mathsf{E}_{\mathsf{P}(X)}(P) \wedge P(u)$.

A.3 The functor $\nabla : \mathsf{Set} \to \mathsf{Eff}$

The topos of sets Set is (equivalent to) the topos of sheaves for the $\neg\neg$-topology on Eff. The direct image part of the inclusion $\mathsf{Set} \to \mathsf{Eff}$ is the functor $\nabla : \mathsf{Set} \to \mathsf{Eff}$ which maps a set S to the object $\nabla S = (S, =_{\nabla S})$ where

$$[x =_{\nabla S} y] = \{n \in \mathbb{N} \mid x = y\}.$$

A map $f : S \to T$ is mapped to the morphism $\nabla f : \nabla S \to \nabla T$ represented by the functional relation

$$(\nabla f)(x, y) = \{n \in \mathbb{N} \mid y = f(x)\} \ .$$

The inverse image part is the global sections functor $\Gamma : \mathsf{Eff} \to \mathsf{Set}$, defined as $\Gamma(X) = \mathsf{Eff}(1, X)$.