**Seminar on Logic - 2018/2019. Exercise of the 20th of March.**

Let $S$ be a set and let $\mu\colon \mathcal{P}(S) \to \{0,1\}$ be an ultrafilter over $S$. Let $\overline{\mu} := \{S_0 \in \mathcal{P}(S) : \mu(S_0) = 1\}$. Let $\{M_s\}_{s\in S}$ be a family of nonempty sets. We define:

*the ultraproduct* $\Pi_{(s\in S)}M_s/\mu$ *of the family* $\{M_s\}_{s\in S}$ *w.r.t. the ultrafilter* $\mu$

as the quotient:

$$(\Pi_{(s\in S)}M_s)/_{\sim}$$

where, for every $f, g \in \Pi_{(s\in S)}M_s$, we say that $f \sim g$ iff $\{s \in S : f(s) = g(s)\} \in \overline{\mu}$.

Prove that the diagram:

$$(\overline{\mu}, \supseteq) \to \textsc{Set}$$
$$(\ S_0 \supseteq S_1\ ) \mapsto (\ (\Pi_{(s\in S_0)}M_s) \ni f \mapsto f\restriction_{S_1} \in (\Pi_{(s\in S_1)}M_s)\ )$$

has the ultraproduct $\Pi_{(s\in S)}M_s/\mu$ as colimit, exhibiting the corresponding arrows $\Pi_{(s\in S_0)}M_s \to \Pi_{(s\in S)}M_s/\mu$.

During the seminar, we used this characterization in order to prove that $\Pi_{(s\in S)}M_s/\delta_{s_0}$ is isomorphic (as a set) to $M_{s_0}$ (for every choice of $s_0 \in S$), being $\delta_{s_0}$ the ultrafilter over $S$ defined by $\delta_{s_0}(S_0) = 1$ iff $S_0 \ni s_0$, for every $S_0 \subseteq S$ (actually we did so in a more general situation that includes this one). However, we can also prove this fact by exhibiting a very natural set-theoretic bijection: find this bijection and enjoy it!