Seminar on Logic. Exercise to be handed in 13th of March

1. (5pt.) Let $\mathcal{M}$ and $\mathcal{N}$ be categories which allow ultraproducts in itself. Let $F : \mathcal{M} \to \mathcal{N}$ be a functor that preserves small filtered colimits. We have seen in the lecture that, for every collection of objects M_s and ultrafilter μ on S there exists a (unique) map σ_μ such that the following diagram commutes:

$$\begin{array}{ccc} F(\prod_{s\in S_0} M_s) & \longrightarrow & \prod_{s\in S_0}(F(M_s)) \\ \downarrow{\scriptstyle F(q_\mu^{S_0})} & & \downarrow{\scriptstyle q_\mu^{S_0}} \\ F(\int_S M_s d\mu) & \xrightarrow{\sigma_\mu} & \int_S (F(M_s)) d\mu \end{array}$$

   Show that the morphisms $\{\sigma_\mu\}$ satisfy conditions (0) and (1) of Definition 1.4.1, that is, show that they can be seen as an ultrastructure on F, so F can be regarded as an ultrafunctor[1].

2. (3pt.) Show that the ultraproduct functor $\int_S (\bullet)\, d\mu$ preserves finite limits, initial objects and effective isomorphisms, as is stated in Proposition 2.1.3.

3. (2pt.) Show that there exists a bijection $\mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Fun}(\mathcal{C}, \mathrm{Set})) \to \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{M}, \mathrm{Set}))$ by giving the bijection explicitly.

[1] For condition (2), I refer to Proposition 1.4.9.