Seminar on Logic - 2018/2019. Exercise of the 20th of March.

Let S be a set and let $\mu\colon \mathcal{P}(S) \to \{0,1\}$ be an ultrafilter over S. Let $\overline{\mu} := \{S_0 \in \mathcal{P}(S) : \mu(S_0) = 1\}$. Let $\{M_s\}_{s\in S}$ be a family of nonempty sets. We define:

the ultraproduct $\Pi_{(s\in S)}M_s/\mu$ *of the family* $\{M_s\}_{s\in S}$ *w.r.t. the ultrafilter* μ

as the quotient:

$$(\Pi_{(s\in S)}M_s)/_\sim$$

where, for every $f, g \in \Pi_{(s\in S)}M_s$, we say that $f \sim g$ iff $\{s \in S : f(s) = g(s)\} \in \overline{\mu}$.

Prove that the diagram:

$$(\overline{\mu}, \supseteq) \to \textsc{Set}$$
$$(\ S_0 \supseteq S_1\ ) \mapsto (\ (\Pi_{(s\in S_0)}M_s) \ni f \mapsto f\restriction_{S_1} \in (\Pi_{(s\in S_1)}M_s)\ )$$

has the ultraproduct $\Pi_{(s\in S)}M_s/\mu$ as colimit, exhibiting the corresponding arrows $\Pi_{(s\in S_0)}M_s \to \Pi_{(s\in S)}M_s/\mu$.

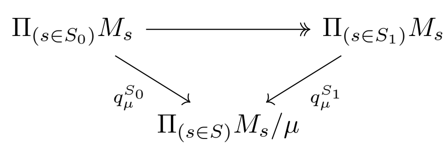
During the seminar, we used this characterization in order to prove that $\Pi_{(s\in S)}M_s/\delta_{s_0}$ is isomorphic (as a set) to M_{s_0} (for every choice of $s_0 \in S$), being δ_{s_0} the ultrafilter over S defined by $\delta_{s_0}(S_0) = 1$ iff $S_0 \ni s_0$, for every $S_0 \subseteq S$ (actually we did so in a more general situation that includes this one). However, we can also prove this fact by exhibiting a very natural set-theoretic bijection: find this bijection and enjoy it!

Solution.

Let $S_0 \in \overline{\mu}$. For every $f \in \Pi_{(s\in S_0)}M_s$ let us pick an element $\hat{f} \in \Pi_{(s\in S)}M_s$ such that $\hat{f}\restriction_{S_0} = f$ (we are using the axiom of choice). Then the map:

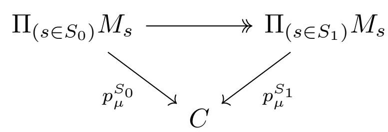
$$q_\mu^{S_0}\colon \Pi_{(s\in S_0)}M_s \ni f \mapsto [\hat{f}] \in \Pi_{(s\in S)}M_s/\mu$$

does not depend on the choice of $\hat{f}$. Indeed, whenever $\hat{f}' \in \Pi_{(s\in S)}M_s$ is such that $\hat{f}'\restriction_{S_0} = f$, it is the case that $\hat{f}'\restriction_{S_0} = \hat{f}\restriction_{S_0}$ and then $\hat{f}' \sim \hat{f}$. Moreover, whenever $S_0 \supseteq S_1$ is an arrow of $(\overline{\mu}, \supseteq)$, it is the case that the following diagram:



commutes and therefore the family $\{q_\mu^{S_0}\}_{(S_0\in\overline{\mu})}$ exhibits $\Pi_{(s\in S)}M_s/\mu$ as a cocone of the given diagram.

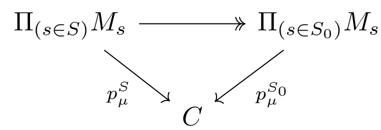
Let C be a cocone of the given diagram, with the family $\{p_\mu^{S_0}\colon \Pi_{(s\in S_0)}M_s \to C\}$. Then, whenever $S_0 \supseteq S_1$ is an arrow of $(\overline{\mu}, \supseteq)$, it is the case that the following diagram:



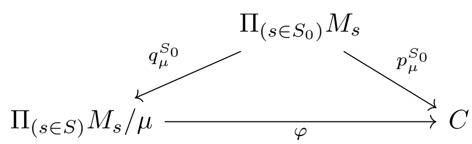
commutes. Then the map:

$$\varphi\colon \Pi_{(s\in S)}M_s/\mu \ni [f] \mapsto p_\mu^S(f) \in C$$

is well-defined: let us assume that $f, g \in \Pi_{(s\in S)}M_s$ are such that $f \sim g$. Then there is $S_0 \in \overline{\mu}$ such that $f \restriction_{S_0} = g \restriction_{S_0}$ and, since the following diagram:



commutes, it is the case that $p_\mu^S(f) = p_\mu^{S_0}(f \restriction_{S_0}) = p_\mu^{S_0}(g \restriction_{S_0}) = p_\mu^S(g)$. Moreover, whenever $S_0 \in \overline{\mu}$, it is the case that the following diagram:



commutes: let $f \in \Pi_{(s\in S_0)}M_s$. Then $q_\mu^{S_0}(f) = [\hat{f}]$ for some $\hat{f} \in \Pi_{(s\in S)}M_s$ such that $\hat{f} \restriction_{S_0} = f$. Then $\varphi(q_\mu^{S_0}(f)) = \varphi([\hat{f}]) = p_\mu^S(\hat{f}) = p_\mu^{S_0}(\hat{f} \restriction_{S_0}) = p_\mu^{S_0}(f)$, where the third equality holds because of the commutativity of the previous diagram. We proved that φ is an arrow between cocones of the given diagram. Hence, being C an arbitary cocone, it is the case that $\Pi_{(s\in S)}M_s/\mu$ is the initial one.

Let $s_0 \in S$. Then $\Pi_{(s\in S)}M_s/\delta_{s_0} \ni [f] \mapsto f(s_0) \in M_{s_0}$ is well-defined: whenever $f, g \in \Pi_{(s\in S)}M_s$ are such that $f \sim g$, it is the case that $f \restriction_{S_0} = g \restriction_{S_0}$ for some $S_0 \in \overline{\delta_{s_0}}$. Since $s_0 \in S_0$, it holds that $f(s_0) = g(s_0)$.

If $[f], [g] \in \Pi_{(s\in S)}M_s/\delta_{s_0}$ are such that $f(s_0) = g(s_0)$, then $f \restriction_{\{s_0\}} = g \restriction_{\{s_0\}}$. Being $\{s_0\} \in \overline{\delta_{s_0}}$, it is the case that $[f] = [g]$. This proves that our map is injective. Moreover, if $m \in M_{s_0}$, the class $[f]$ of an element $f \in \Pi_{(s\in S)}M_s$ such that $f(s_0) = m$ (we are using the axiom of choice) is sent to m, and this proves that our map is surjective.