# Seminar on Logic. Exercise to be handed in 13th of March

1. We first look at condition (0). There are some commutative diagrams that have to be constructed, but once you have that, the rest follows quite easily. The first is already given in the exercise:

$$\begin{array}{ccc} F(\prod_{s\in S_0} M_s) & \xrightarrow{P} & \prod_{s\in S_0}(F(M_s)) \\ \downarrow{\scriptstyle F(q_\mu^{S_0})} & & \downarrow{\scriptstyle q_\mu^{S_0}} \\ F(\int_S M_s d\mu) & \xrightarrow{\sigma_\mu} & \int_S (F(M_s))d\mu \end{array}$$

Furthermore, by definition of the $q_\mu^{S_0}$, we have the following commuting diagram:

$$\begin{array}{ccc} \prod_{s\in S_0} M_s & \xrightarrow{\prod_{s\in S_0} f_s} & \prod_{s\in S_0} M'_s \\ {\scriptstyle q_\mu^{S_0}}\downarrow & & \downarrow{\scriptstyle q_\mu^{S_0}} \\ \int_S M_s d\mu & \xrightarrow{\int_S f_s d\mu} & \int_S M'_s d\mu \end{array}$$

And of course we have a similar diagram for $F(q_\mu^{S_0})$. We can also see that the following diagram commutes:

$$\begin{array}{ccc} F(\prod_{s\in S_0} M_s) & \xrightarrow{F(\prod_{s\in S_0} f_s)} & F(\prod_{s\in S_0} M'_s) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P'} \\ \prod_{s\in S_0} F(M_s) & \xrightarrow{\prod_{s\in S_0} F(f_s)} & \prod_{s\in S_0} F(M'_s) \end{array}$$

This is because both the clockwise and the counter-clockwise are the same at every component; that is easy to check. Now in the same way as above, we also have the following commutative diagrams:

$$\begin{array}{ccc} \prod_{s\in S_0} F(M_s) & \xrightarrow{\prod_{s\in S_0} F(f_s)} & \prod_{s\in S_0} F(M'_s) \\ \downarrow{\scriptstyle q_\mu^{S_0}} & & \downarrow{\scriptstyle q_\mu^{S_0}} \\ \int_S F(M_s)d\mu & \xrightarrow{\int_S F(f_s)d\mu} & \int_S F(M'_s)d\mu \end{array}$$

$$\begin{array}{ccc} F(\prod_{s\in S_0} M'_s) & \xrightarrow{P'} & \prod_{s\in S_0}(F(M'_s)) \\ \downarrow{\scriptstyle F(q_\mu^{S_0})} & & \downarrow{\scriptstyle q_\mu^{S_0}} \\ F(\int_S M'_s d\mu) & \xrightarrow{\sigma_\mu} & \int_S (F(M'_s))d\mu \end{array}$$

Now from all of these diagrams, we can conclude the following equivalence: $\sigma_\mu \circ F(\int_S f_s d\mu) \circ F(q_\mu^{S_0}) = \sigma_\mu \circ F(q_\mu^{S_0}) \circ F(\prod_{s\in S_0} f_s) = q_\mu^{S_0} \circ P' \circ F(\prod_{s\in S_0} f_s) = q_\mu^{S_0} \circ \prod_{s\in S_0} F(f_s) \circ P = \int_S F(f_s)d\mu \circ q_\mu^{S_0} \circ P = \int_S F(f_s)d\mu \circ \sigma_\mu \circ F(q_\mu^{S_0})$. Now we use that $F$ preserves filtered colimits to conclude that $\sigma_\mu \circ F(\int_S f_s d\mu) = \int_S F(f_s)d\mu \circ \sigma_\mu$. So that is condition (0).

For condition (1) we look at the diagram of the exercise again. We take $\mu = \delta_{s_0}$ for some $s_0 \in S$ and we take $S_0 = \{s_0\}$, so that we have $\mu(S_0) = 1$. This gives us that $F(\prod_{s\in S_0} M_s) = F(M_{s_0}) = \prod_{s\in S_0} F(M_s)$. So the map $P$ is the identity in this special case.

Furthermore, the map $\epsilon_{S,s_0}$ is special in this particular case. Let $f_{s_0} : M_{s_0} \to M'_{s_0}$ be given. Then we see that the following diagram commutes:

$$\begin{array}{ccc} \int_S M_{s_0} d\delta_{s_0} & \xrightarrow{\int_S f_{s_0} d\delta_{s_0}} & \int_S M'_{s_0} d\delta_{s_0} \\ {\scriptstyle \epsilon_{S,s_0}} \downarrow & & \downarrow {\scriptstyle \epsilon_{S,s_0}} \\ M_{s_0} & \xrightarrow{f_{s_0}} & M'_{s_0} \end{array}$$

From this, and the definition of $q_\mu^{\{s_0\}}$, we may conclude that $q_\mu^{\{s_0\}} = \epsilon_{S,s_0}^{-1}$. Using this in combination with the fact of map $P$, we get that $\sigma_{\delta_{s_0}} \circ F(\epsilon_{S,s_0}^{-1}) = \epsilon_{S,s_0}^{-1}$. Rewriting this, we get that $\epsilon_{S,s_0} \circ \sigma_{\delta_{s_0}} = F(\epsilon_{S,s_0})$. And this is exactly condition (1).

2. The first important fact to notice is that in Set filtered colimits commute with finite limits. One can find this fact, for example, in the Elephant. Using this fact, we can see that we only need to bother with the functors of the form $\{M_s\}_{s \in S} \to \prod_{s \in S_0} M_s$. So we show that these functors preserve finite limits, initial object and effective epis.

   We start with the case of finite limits. Since we are talking about a product functor, it is trivial to see that the initial object and binary products are preserved. A little set theoretical argument then also gives that equalizers are preserved.

   The initial object in $\mathrm{Set}^S$ is the constant presheaf 0. Since the product over initial objects is clearly initial, we see that initial objects are also preserved.

   For the effective epis, we need another observation. Since Set is a regular category, all epis are effective. So instead of looking at effective epis, we can, and have to, look at epis in general. So look at an epi $f$ in $\mathrm{Set}^S$. If $f$ is an epi, that means that every one of its components is a surjective function. This means that if you take the product over the components of $f$, you still get a surjective function, which means epi, and therefore also effective epi. This gives that the functor also preserves effective epis.

3. First we note that it is easy to see that there is a bijection if we ignore the fact that we are looking at left-ultrafunctors, specifically. That is, for any element $F \in \mathrm{Fun}(\mathcal{M}, \mathrm{Fun}(\mathcal{C}, \mathrm{Set}))$, we can define a functor $G \in \mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{M}, \mathrm{Set}))$ as follows: $G(C)(M) = F(M)(C)$. One can easily see that this is a bijection.

   Now the question is as follows: suppose we have that $F \in \mathrm{Fun}(\mathcal{M}, \mathrm{Fun}^{\mathrm{LUlt}}(\mathcal{C}, \mathrm{Set}))$, can we use the same bijection? That is, can we define an ultrastructure on $G$? We already have an ultrastructure $\sigma_\mu : F(M)(\int_S C_s d\mu) \to \int_S F(M)(C_s) d\mu$. So we look at the following:

$$G(\int_S C_s d\mu)(M) = F(M)(\int_S C_s d\mu) \to \int_S F(M)(C_s) d\mu = \int_S G(C_s)(M) d\mu$$

   Here the arrow is the $\sigma_\mu$ of $F$. So we see that we indeed have a very natural transformation $G(\int_S C_s d\mu)(M) \to \int_S G(C_s)(M) d\mu$. It then is little work to show that this is a bijection, because the inverse function is easily shown to have the same properties.