Seminar Ultracategories - Model solution 5

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Let P be a complete linear order, i.e. a linear order such that all subsets of P have a least upper and greatest lower bound. Note that P, as a poset category, is then also complete, hence it has a categorical ultrastructure.

1. Show that the category Stone_P is equivalent to the category with:

- **Objects (X, f), where X is a Stone space and $f : X \to P$ is a function such that for all μ there is an $S_0 \in \mu$ and $s \in S_0$ with $f(\int_{s\in S} x_s \,\mathrm{d}\mu) \leq f(x_s)$;**

- **Arrows $(X, f) \xrightarrow{\phi} (Y, g)$ are continuous functions $\phi : Y \to X$ such that for all $y \in Y$, $g(y) \leq f(\phi(y))$.**

By definition, an object of Stone_P is a Stone space X, together with an ultrafunctor $f : X \to P$. We will show that a map $f : X \to P$ is an ultrafunctor if and only if for all ultrafilters μ there is an s with $f(\int_{s\in S} x_s \,\mathrm{d}\mu) \leq f(x_s)$. Since all diagrams in P commute and there are no nontrivial arrows in X, f is an ultrafunctor if and only if for all $\{x_s\}_{s\in S}$ we have $f(\int_{s\in S} x_s \,\mathrm{d}\mu) \leq \int_{s\in S} f(x_s) \,\mathrm{d}\mu$. Thus, we need to check that the given condition is equivalent to $f(\int_{s\in S} x_s \,\mathrm{d}\mu) \leq \int_{s\in S} f(x_s) \,\mathrm{d}\mu$. In the case that P is finite, we have $f(\int_{s\in S} x_s \,\mathrm{d}\mu) \leq \int_{s\in S} f(x_s) \,\mathrm{d}\mu$ for all $x : S \to X$ and $\mu : \beta S$, then we have $\int_{s\in S} f(x_s) \,\mathrm{d}\mu = \max_{S_0 \in \mu} \min_{s\in S_0} f(x_s)$, so we can simply take the corresponding S_0 and s, and get the equivalence.

Now we need to extend this to an arbitrary linear order P. Given $\mu \in \beta S$ and $x : S \to X$, and let f be an ultrafunctor. We want to show there is an $s \in S$ with $f(\int_{s\in S} x_s \,\mathrm{d}\mu) \leq f(x_s)$. Since the principal ultrafilters are dense in βS, we can write μ as the limit of principal ultrafilters $\{\delta_n\}$. Consider the map $\int_{s\in S} x_s \,\mathrm{d}\_ : \beta S \to X$. This is a continuous map between compact Hausdorff spaces, so its image is closed. Let $\{x_n\}$ be the sequence given by applying this map to each δ_n, then this will also converge to $\int_{s\in S} x_s \,\mathrm{d}\mu$. Moreover, for each n we have $f(\int_{s\in S} x_s \,\mathrm{d}\delta_n) \leq f(x_n)$ since f is an ultrafunctor. Note that it does not really make sense to take the limit of $f(x_n)$ since that would mix the concept of topological limit with the concept of limit in orders. We can fix this by considering the limit supremum, $\limsup_{n\in\mathbb{N}} x_n$ (since the topology on P for which limits coincide with suprema is the topology of upsets A_P from the next exercise). Let y_n be the subsequence (allowing repetitions) of x_n occuring in the limit supremum, then it is a subsequence of a converging sequence in a closed subspace of a compact Hausdorff space. Thus, it converges to some point of this closed subspace (not necessarily the same if y_n remains constant for large enough n), which is exactly the x_s that we are looking for (and we can take S_0 to be S itself).

For the converse, suppose $f(\int_{s\in S} x_s \,\mathrm{d}\mu) > \int_{s\in S} f(x_s) \,\mathrm{d}\mu$. Since the ultraproduct is a supremum over the elements of μ, we also have $f(\int_{s\in S} x_s \,\mathrm{d}\mu) > \prod_{s\in S_0} f(x_s)$ for all $S_0 \in \mu$. The product in a poset is the greatest lower bound, and we have a strict inequality, we have $f(\int_{s\in S} x_s \,\mathrm{d}\mu) > f(x_s)$ for $x_s \in S_0$, which is exactly the negation of the condition $f(\int_{s\in S} x_s \,\mathrm{d}\mu) \leq f(x_s)$.

For arrows, the conditions for $(\phi, \alpha) : (X, f) \to (Y, g)$ in Stone_P state that $\phi : Y \to X$ is a continuous function, and that α is a natural transformation of ultrafunctors $g \Rightarrow f \circ \phi$, which is just a natural transformation that commutes with the ultrastructures on the two functors. Since Stone spaces have only trivial arrows and all diagrams in P commute, this means that at each object $y \in Y$ we have an arrow from $g(y)$ to $f(\phi(y))$. This is exactly the condition that we require.

2. Let A_P be the topology on P where a set $U \subseteq P$ is open iff it is an upset (i.e. $x \in U$ and $x \leq y$ imply $y \in U$). Show that $\mathsf{Shv}(A_P)$ is equivalent to $\mathrm{Fun}(P, \mathsf{Set})$.

Let $\mathcal{F} : A_P^{\mathrm{op}} \to \mathsf{Set}$ be a sheaf, then we can define a functor $F : P \to \mathsf{Set}$ as follows: Given an object $x \in P$, define $\uparrow(x) = \{y \mid y \geq x\}$, and note that this is indeed open in A_P. We set $F(x) = \mathcal{F}(\uparrow(x))$. Given an arrow $x \leq y$ in P, by transitivity of the order we have that $\uparrow(y) \subseteq \uparrow(x)$. Applying $\mathcal{F}$ to this inclusion gives an arrow in the opposite direction $F(x) = \mathcal{F}(\uparrow(x)) \to \mathcal{F}(\uparrow(y)) = F(y)$. Note that $\mathcal{F}$ being a presheaf implies that F is a functor. Any natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ will give rise to a natural transformation of the corresponding functor, since the functor is essentially a restriction of the sheaf.

For the opposite direction, let $F : P \to \mathsf{Set}$ be a functor, then we want to find a sheaf $\mathcal{F} : A_P^{\mathrm{op}} \to \mathsf{Set}$. Note that a basis of A_P is given by all the upsets of the form $\uparrow(x) = \{y \mid y \geq x\}$, together with the empty set, since any upset U is equal to $\bigcup_{x \in U} \uparrow(x)$. Thus, it suffices to specify $\mathcal{F}$ on this basis to get a sheaf up to natural isomorphism, so we set $\mathcal{F}(\emptyset) = \top$ for some choice of terminal object $\top$, and $\mathcal{F}(\uparrow(x)) = F(x)$. For the arrows to $\mathcal{F}(\emptyset)$ we take the unique maps to the terminal object (which automatically satisfy the sheaf axioms), and for other arrows $\mathcal{F}(\uparrow(x)) \to \mathcal{F}(\uparrow(y))$ for $\uparrow(y) \subseteq \uparrow(x)$, we note that $y \in \uparrow(y) \subseteq \uparrow(x)$ implies $x \leq y$, so we can take the arrow $F(x \leq y) : F(x) = \mathcal{F}(\uparrow(x)) \to \mathcal{F}(\uparrow(y)) = F(y)$. This gives a presheaf since we can just extend the proof that F is a functor with the fact that the maps to the terminal object $\top$ are unique. Moreover $\mathcal{F}$ is a sheaf, since any open cover of a basic open $\uparrow(x)$ must contain $\uparrow(x)$ itself, so the locality and gluing axioms are automatically satisfied. By uniquely extending a given natural transformation $F \Rightarrow G$, we find a map of sheaves $\mathcal{F} \Rightarrow \mathcal{G}$.

It remains to show that these constructions are pseudo-inverses to each other. If we start with a functor $F : P \to \mathsf{Set}$, construct a sheaf, then go back to a functor, we get exactly the same functor back. On the other hand, if we start with a sheaf $\mathcal{F}$, then apart from $\emptyset$ we restrict $\mathcal{F}$ to the basis consisting of sets of the form $\uparrow(x)$, and then extend it uniquely, up to natural isomorphism, back to $\mathcal{F}'$. This new sheaf is isomorphic to $\mathcal{F}$ except perhaps in its component at $\emptyset$. However, by the sheaf axioms, since $\emptyset$ is an initial object, we have that $\mathcal{F}(\emptyset)$ is terminal (and $\mathcal{F}'(\emptyset)$ is terminal by construction). Thus we can find an isomorphism of sheaves from $\mathcal{F}$ to $\mathcal{F}'$, which sends one terminal object to another, and leaves other components in place.

3. Show that all functors in $\mathrm{Fun}(P, \mathsf{Set})$ have a left ultrastructure.

Here I will use a nice argument given by Bart and Matteo. Given $F : P \to \mathsf{Set}$, we find a left ultrastructure on F using Proposition 1.4.9. Suppose we have $M : S \to P$ and $\mu : \beta S$, we want to show that the maps $F(q_\mu^{S_0}) : F(\prod_{s \in S_0} M_s) \to F(\int_{s \in S} M_s \, \mathrm{d}\mu)$ exhibit $F(\int_{s \in S} M_s \, \mathrm{d}\mu)$ as the colimit of the diagram consisting of all $F(\prod_{s \in S_0} M_s)$. Note that in P, we have that $q_\mu^{S_0}$ is just the inequality $\prod_{s \in S_0} M_s \leq \int_{s \in S} M_s \, \mathrm{d}\mu$.

Sending this through the equivalence of the previous exercise, we find a sheaf $\mathcal{F}$ on A_P, and we want to know whether $\mathcal{F}(\uparrow(\int_{s \in S} M_s \, \mathrm{d}\mu) \subseteq \uparrow(\prod_{s \in S_0} M_s))$ form the colimiting cocone over the diagram of all $\mathcal{F}(\uparrow(\prod_{s \in S_0} M_s))$ for $S_0 \in \mu$. Since the inclusion between sets $\uparrow(x)$ of A_P just form the dual category to P itself, and by completeness of P, colimits in P are sent to limits and A_P, we have that $\uparrow(\int_{s \in S} M_s \, \mathrm{d}\mu)$ is the limit of the diagram $\uparrow(\prod_{s \in S_0} M_s)$ for $S_0 \in \mu$. Using the fact that sheaves send limits in A_P to colimits in Set, we conclude that $F(q_\mu^{S_0})$ exhibits $F(\int_{s \in S} M_s \, \mathrm{d}\mu)$ as the colimit of the diagram consisting of all $F(\prod_{s \in S_0} M_s)$, so F has an ultrastructure.

4. Can we conclude that $\mathrm{Fun}^{\mathrm{LUlt}}(P, \mathsf{Set})$ is equivalent to $\mathsf{Shv}(A_P)$?

This is the case if and only if each functor $F : P \to \mathsf{Set}$ has a left ultrastructure that is unique up to natural isomorphism of left ultrafunctors. By Proposition 1.4.9 and the result of the previous exercise, this happens exactly when all ultrafunctors F, σ_μ have the property that for all $\mu \in \beta S$ and $S_0 \in \mu$, the following diagram commutes up to natural isomorphism of σ_μ:

$$\begin{array}{ccc} F(\prod_{s\in S_0} M_s) & \longrightarrow & \prod_{s\in S_0} F(M_s) \\ \big\downarrow{\scriptstyle F(q_\mu^{S_0})} & & \big\downarrow{\scriptstyle q_\mu^{S_0}} \\ F(\int_{s\in S} M_s \,\mathrm{d}\mu) & \xrightarrow{\sigma_\mu} & \int_{s\in S} F(M_s)\,\mathrm{d}\mu. \end{array}$$

Following the previous exercise, we can assume that F arises from restricting a sheaf $\mathcal{F}$ on A_P to the basis that corresponds to P itself, and again use that sheaves send limits in A_P to colimits in Set, we want to know whether the following diagram commutes up to natural isomorphism of σ_μ:

$$\begin{array}{ccc} \mathcal{F}(\uparrow(\int_{s\in S} M_s \,\mathrm{d}\mu)) & \xrightarrow{\sigma_\mu} & \varinjlim_{S_0\in\mu} \prod_{s\in S_0} \mathcal{F}(\uparrow(M_s)) \\ \big\downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle \varinjlim_{S_0\in\mu}\phi_{S_0}} & \\ \varinjlim_{S_0\in\mu} \mathcal{F}(\uparrow(\prod_{s\in S_0} M_s)) & & \end{array},$$

where $\phi_{S_0} : \mathcal{F}(\uparrow(\prod_{s\in S_0} M_s)) = F(\prod_{s\in S_0} M_s) \to \prod_{s\in S_0} F(M_s) = \prod_{s\in S_0} \mathcal{F}(\uparrow(M_s))$ is the canonical map switching the order of functor and product.

First, we will show the diagram commutes for principal ultrafilters $\mu = \delta_s$. In that case, we have that $\int_{s\in S} M_s \,\mathrm{d}\delta_s = M_s$, so we have the following diagram:

$$\begin{array}{ccc} \mathcal{F}(\uparrow(M_s)) = \mathcal{F}(\uparrow(\int_{s\in S} M_s \,\mathrm{d}\delta_s)) & \xrightarrow{\sigma_{\delta_s}} & \int_{s\in S} M_s \,\mathrm{d}\delta_s \\ {\scriptstyle \mathsf{id}=F(\epsilon_s)}\big\downarrow & \swarrow{\scriptstyle \epsilon_s} & \\ \mathcal{F}(\uparrow(M_s)) & & \end{array},$$

which commutes due to the ultrastructure axioms.

Since ultraproducts are dense in βS, write an arbitrary ultraproduct μ as $\mu = \int_{s\in S} \delta_s \,\mathrm{d}\mu$. Next, we extend S by adjoining a new element $*$, and set $m_* = \prod_{s\in S_0} M_s$. The inclusion $S \hookrightarrow S \cup \{*\}$ gives rise to a monomorphism $\beta S \hookrightarrow \beta(S \cup \{*\})$, where the image of a principal ultrafilter δ_s is just the corresponding principal ultrafilter δ_s (on a larger powerset), while the image of μ is $\mu_* = \int_{s\in S} \delta'_s \,\mathrm{d}\mu$ (with δ'_s a principal ultrafilter on $S \cup \{*\}$). We extend a given $M : S \to P$ to $S \cup \{*\}$ by setting $M'_* = \prod_{s\in S_0} M_s$. Since it is a principal ultrafilter, σ_{δ_*} makes the diagram for M' commute. By construction, we also have that σ_{μ_*} makes the following diagram commute, for S_0 specifically:

$$\begin{array}{ccc} F(\prod_{s\in S_0} M_s) = F(M'_*) & \longrightarrow & \prod_{s\in S_0} F(M_s) \\ \big\downarrow{\scriptstyle F(q_{\mu_*}^{S_0})} & & \big\downarrow{\scriptstyle q_{\mu_*}^{S_0}} \\ F(\int_{s\in S} M_s \,\mathrm{d}\mu_*) & \xrightarrow{\sigma_{\mu_*}} & \int_{s\in S} F(M_s)\,\mathrm{d}\mu_*. \end{array}$$

(where we use that M' restricted to a subset of S is just M). Applying the Fubini transform to $\mu = \int_{s\in S} \delta_s \,\mathrm{d}\mu$ and $\mu_* = \int_{s\in S} \delta'_s \,\mathrm{d}\mu$, we find that this defines σ_μ up to natural isomorphism as σ_{μ_*}, which was unique by Proposition 1.4.9.

Thus, we conclude that the left ultrastructures on a functor $F : P \to \mathsf{Set}$ are unique up to isomorphism, which means that $\mathrm{Fun}^{\mathrm{LUlt}}(P, \mathsf{Set})$ is indeed equivalent to $\mathsf{Shv}(A_P)$.