Ultracategories seminar # 6 : Exercises

For a total of 10 points. To be submitted by April 17, 2019.

Solutions

1. On this sheet of paper, we say a category $\mathscr{C}$ is *filtered* if for each pair of objects C, D in $\mathscr{C}$ there is an object E with maps $C \to E$ and $D \to E$, and for parallel maps $f, g\colon C \rightrightarrows D$ there is an object E with a map $h\colon C \to E$ such that $hf = hg$. We say $\mathscr{C}$ is *cofiltered* if $\mathscr{C}^{\mathrm{op}}$ is filtered. Show that filtered colimits commute with finite limits in Set. (1 point)

   Let $\mathscr{D}$ be a filtered category, and $F, G\colon \mathscr{D} \to \mathrm{Set}$ two $\mathscr{D}$-shaped diagrams in Set. Then we have natural isomorphisms in Set

   $$\operatorname*{colim}_{\mathscr{D}} F \times \operatorname*{colim}_{\mathscr{D}} G \simeq \operatorname*{colim}_{\mathscr{D}}(\operatorname*{colim}_{\mathscr{D}} F) \times G \simeq \operatorname*{colim}_{\mathscr{D}\times\mathscr{D}} (F \times G) \simeq \operatorname*{colim}_{\mathscr{D}} (F \times G).$$

   The first isomorphism follows from the fact Set is cartesian closed, so the bifunctor $- \times -$ preserves small colimits in each variable, the second follows from the "Fubini theorem" for colimits, and the third from the fact the diagonal functor $\mathscr{D} \to \mathscr{D} \times \mathscr{D}$ is final. For the equaliser case we'll be more explicit. Let X_d, Y_d be $\mathscr{D}$-shaped diagrams in Set and let $f, g\colon X_d \to Y_d$ be two morphisms of $\mathscr{D}$-shaped diagrams, i.e., natural transformations of functors. We then have a natural map

   $$\Theta\colon \operatorname*{colim}_{\mathscr{D}} \mathrm{eq}(X_d \rightrightarrows Y_d) \longrightarrow \mathrm{eq}\left(\operatorname*{colim}_{\mathscr{D}} X_d \rightrightarrows \operatorname*{colim}_{\mathscr{D}} Y_d\right), \qquad x \longmapsto x.$$

   Let x, y be two elements in the domain of Θ, which, by the filteredness of $\mathscr{D}$ we may without loss of generality represent by elements $x, y \in X_d$. For injectivity, suppose $\Theta(x) = \Theta(y)$, then they must agree in $\mathrm{colim} X_d$ and hence there must be a z in X_e, and $h\colon d \to e$ in $\mathscr{D}$ such that $X(h)(x) = X(h)(y)$. However, this condition implies x and y are also equal in the domain of Θ. For surjectivity, take some z in the codomain of Θ. As z belongs to this codomain we know $f(z) = g(z)$, in other words, we can represent z by an element of X_d such that $f_d(z) = g_d(z)$ using the fact $\mathscr{D}$ is filtered. It is then clear that z is in the image of Θ.

2. Recall that functors $F\colon \mathscr{C} \to \mathrm{Set}$ are naturally isomorphic to a colimit of representables.

   (a) Let $\mathscr{C}$ be a category with finite limits. Show a functor $F\colon \mathscr{C} \to \mathrm{Set}$ is naturally isomorphic to a filtered colimit of representables if and only if F preserves finite limits. (2 points)

If F is a filtered colimits of representables, then, as representables commute with small limits, (co)limits in functor categories are calculated pointwise, and filtered colimits in Set commute with finite limits, then F preserves finite limits.

Conversely, if F preserves finite limits, we write F as the following colimit of representables,

$$F \simeq \operatorname*{colim}_{\int_{\mathscr{C}^{\mathrm{op}}} F} \mathrm{Hom}_{\mathscr{C}}(C, -).$$

To see this category of elements $\int_{\mathscr{C}^{\mathrm{op}}} F$ is filtered, we will show $(\int_{\mathscr{C}^{\mathrm{op}}} F)^{\mathrm{op}} \simeq \int_{\mathscr{C}} F$ is cofiltered. Consider objects (C, x) and (C', x') in $\int_{\mathscr{C}} F$. There is an object $C \times C'$ in $\mathscr{C}$ and as F preserves finite limits we have a bijection of sets $\phi\colon F(C) \times F(C') \to F(C \times C')$. The object $(C \times C', \phi(x, x'))$ is then common to (C, x) and (C', x'). Let $f, g\colon (C, x) \rightrightarrows (C', x')$ be parallel arrows in $\mathscr{C}$. Let C'' be the equaliser of f and g in $\mathscr{C}$, equipped with a canonical map $h\colon C'' \to C$. As F preserves finite limits, then $F(C'')$ equalises the maps $F(f)$ and $F(g)$ in Set, so in particular we can consider $x \in F(C'') \subseteq F(C)$. The object (C'', x) then maps to (C, x) in $\int_{\mathscr{C}} F$ and the map h satisfies $hf = hg$.

(b) Show the Yoneda embedding $\mathscr{C} \to \mathrm{Pro}(\mathscr{C})$ commutes with finite limits and all small colimits that exist in $\mathscr{C}$. (1 point)

The fact that the Yoneda embedding commutes with all small colimits is clear as $C \mapsto \mathrm{Hom}_{\mathscr{C}}(C, -)$ sends colimits in $\mathscr{C}$ to limits in $\mathrm{Fun}(\mathscr{C}, \mathrm{Set})$. If C_α is a finite diagram in $\mathscr{C}$, then for any $X \in \mathrm{Pro}(\mathscr{C})$ we have the following chain of natural equivalences,

$$\mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, \mathrm{Hom}_{\mathscr{C}}(\lim C_\alpha, -)) \simeq X(\lim C_\alpha) \simeq \lim X(C_\alpha) \simeq \lim \mathrm{Hom}_{\mathrm{Pro}(\mathscr{C})}(X, C_\alpha).$$

3. Let Stone be the category of Stone spaces (and continuous maps), Bool the category of Boolean algebras, Fin the category of finite sets, and Fin Bool the category of finite Boolean algebras.

(a) Show that every Boolean algebra is a filtered colimit of finite Boolean algebras. (1 point)

Let A be a Boolean algebra and $\{A_i\}$ be the filtered diagram of finite Boolean subalgebras of A (this diagram is indeed a filtered poset as inclusions are unique and two finite Boolean subalgebras A_i and A_j of A have a common finite Boolean subalgebra of A generated by the elements of A_i and A_j). We claim the natural map $\omega\colon \mathrm{colim}_i A_i \to A$ induced by the inclusions is an isomorphism of Boolean algebras. Indeed, this map is surjective as for each element $a \in A$ the Boolean subalgebra generated by a is finite, $A_a = \{a, \neg a, 0, 1\}$, hence a is hit by ω. For injectivity, take elements $a \in A_i$ and $b \in A_j$ such that $\omega(a) = \omega(b)$. Using the fact our diagram is filtered we can assume, without loss of generality, that $i = j$, and then injectivity follows as the map $A_i \to A$ is injective, being the defining inclusion.

Let $\mathscr{C}$ be a category with finite colimits. Define $\mathrm{Ind}(\mathscr{C}) = \mathrm{Pro}(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}$. In part (a) we have essentially shown that $\mathrm{Ind}(\mathrm{Fin\,Bool}) \simeq \mathrm{Bool}$.

(b) Use Stone duality to show that $\mathrm{Pro}(\mathrm{Fin}) \simeq \mathrm{Stone}$. (2 points)

Observe the following chain of natural equivalences,

$$\mathrm{Stone} \simeq \mathrm{Bool}^{\mathrm{op}} \simeq (\mathrm{Ind}(\mathrm{Fin\,Bool}))^{\mathrm{op}} = \mathrm{Pro}((\mathrm{Fin\,Bool})^{\mathrm{op}}) \simeq \mathrm{Pro}(\mathrm{Fin\,Stone}) \simeq \mathrm{Pro}(\mathrm{Fin}).$$

The first equivalence is Stone duality, the second is part (a), the third is a definition, the forth is a finite analogue of Stone duality, and the fifth the obvious fact that finite Stone spaces are simply finite sets with a discrete topology.

(c) Let $f\colon X \to Y$ be a continuous map of Stone spaces. Show that f is a homeomorphism if and only if for all finite sets S, the induced map

$$\mathrm{Hom}_{\mathrm{Top}}(Y, S) \xrightarrow{f^*} \mathrm{Hom}_{\mathrm{Top}}(X, S)$$

is a bijection. (1 point)

If f is a homeomorphism then the conclusion is clear. Conversely, let $f\colon X \to Y$ be a map of Stone spaces and S a finite set. Write $\Phi\colon \mathrm{Stone} \to \mathrm{Pro}(\mathrm{Fin})$ for the equivalence of categories of part (b), and Ψ for its pseudo-inverse. From part (b) we notice that $\Psi(S)$ is precisely S considered as a space with the discrete topology. With these observations in place, we notice we have the following natural equivalences,

$$\mathrm{Hom}_{\mathrm{Top}}(Y, S) \simeq \mathrm{Hom}_{\mathrm{Pro(Fin)}}(\Phi Y, S) \simeq \Phi(Y)(S),$$

which sends the induced map f^* to $\Phi(f)\colon \Phi(Y)(S) \longrightarrow \Phi(X)(S)$. The fact this natural map is an isomorphism for all finite sets S implies $\Phi(Y)$ and $\Phi(X)$ are isomorphic as elements in $\mathrm{Pro}(\mathcal{C})$, hence f is an isomorphism in Stone.

4. Let $\mathscr{C}$ be a pretopos. Show the functor of Construction 6.4.3 of the paper, defined by $g\colon \mathrm{Fin} \to \mathscr{C}$ sending $S \mapsto \coprod_S \mathbf{1}$, where $\mathbf{1}$ is the terminal object of $\mathscr{C}$, is left-exact. (2 points)

For finite sets S and T we have natural isomorphisms

$$\left(\coprod_S \mathbf{1}\right) \times \left(\coprod_T \mathbf{1}\right) \simeq \coprod_T \left(\coprod_S \mathbf{1} \times \mathbf{1}\right) \simeq \coprod_{S \times T} \mathbf{1},$$

as $\mathscr{C}$ is cartesian closed, $\mathbf{1} \times \mathbf{1} \simeq \mathbf{1}$, and the "Fubini" theorem for colimits (in particular coproducts). I thought this exercise would involve more work, but the equaliser case follows from condition E_2 of an extensive category, i.e., that finite coproducts is $\mathscr{C}$ commute with pullbacks.