# Hand-in 2

Course: Seminar Logic - Categorical Logic

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This hand-in consists of three exercises.

**Exercise 1.** (5 points) Let $M$ be an interpretation of some language $\mathcal{L}(S)$ of signature $S$. Then for $t_1, t_2$ terms of type $Y$ with free variables among $\overline{z} : \overline{Z}$ we have that $\{\overline{z} | t_1 = t_2\}^{(M)}$ is represented by the equalizer of

$$\overline{Z}^{(M)} \underset{t_2^{(M)}}{\overset{t_1^{(M)}}{\rightrightarrows}} Y^{(M)} .$$

Show that more generally for $\overline{t_1} = (t_{1,1}, t_{1,2}, ..., t_{1,n}), \overline{t_2} = (t_{2,1}, t_{2,2}, ..., t_{2,n})$ finite tupples of terms with free variables among $\overline{z} : \overline{Z}$ such that $t_{i,j}$ is of type $Y_j$ that $\{\overline{z} | \overline{t_1} = \overline{t_2}\}^{(M)}$ is represented by the equalizer of

$$\overline{Z}^{(M)} \underset{\langle t_{2,1}^{(M)}, ..., t_{2,n}^{(M)} \rangle}{\overset{\langle t_{1,1}^{(M)}, ..., t_{1,n}^{(M)} \rangle}{\rightrightarrows}} \overline{Y}^{(M)} .$$

Here $\overline{t_1} = \overline{t_2}$ stands for $\bigwedge_i^n (t_{1,i} = t_{2,i})$ and $\overline{Y}^{(M)} = Y_1^{(M)} \times Y_2^{(M)} \times ... \times Y_n^{(M)}$.

*Solution.* By induction it suffices to prove the following statement: For $\varphi, \psi$ formulas with free variables among $\overline{z} : \overline{Z}$ and arrows $f_1, g_1 : \overline{Z} \to Y_1, f_2, g_2 : X \to Y_2$ such that $\{\overline{z} | \varphi\}$ is represented by the equalizer of $f_1, g_1$ and $\{\overline{z} | \psi\}$ by the equalizer of $f_2, g_2$ then $\{\overline{z} | \varphi \wedge \psi\}$ is given by the equalizer of $\langle f_1, f_2 \rangle$ and $\langle g_1, g_2 \rangle$.

Let $\varphi, \psi, f_1, f_2, g_1, g_2$ be as in the hypothesis above. Then we have the following diagram

$$\begin{array}{ccccc}
\{\overline{z} | \varphi \wedge \psi\}^{(M)} & \xrightarrow{p_2} & \{\overline{z} | \psi\}^{(M)} & & \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle e_2} & & \\
\{\overline{z} | \varphi\}^{(M)} & \xrightarrow[e_1]{} & \overline{Z}^{(M)} & \underset{g_1}{\overset{f_1}{\rightrightarrows}} & Y_1 \\
& & {\scriptstyle f_2}\downarrow\downarrow{\scriptstyle g_2} & & \\
& & Y_2 & &
\end{array}$$

where the square is a pullback square. We defne $e = e_1 \circ p_1 = e_2 \circ p_2$ and show that $e$ is the equalizer of $\langle f_1, f_2 \rangle$ and $\langle g_1, g_2 \rangle$. It is clear from $e_1, e_2$ being equalizers that $e$ makes $\langle f_1, f_2 \rangle, \langle g_1, g_2 \rangle$ equal. It remains to show that $e$ has the equalizer property. Let $e' : E \to \overline{Z}^{(M)}$ also make $\langle f_1, f_2 \rangle, \langle g_1, g_2 \rangle$. Then we have

$$f_1 \circ e' = \pi_1 \circ \langle f_1, f_2 \rangle \circ e' = p_1 \circ \langle g_1, g_2 \rangle \circ e' = g_1 \circ e'$$

such that by the equalizer property for $e_1$ we get unique $h_1 : E \to \{\overline{z} | \varphi\}$ such that $e' = e_1 \circ h_1$. Similarly we get unique $h_2 : E \to \{\overline{z} | \psi\}$ such that $e' = e_1 \circ h_2$. In particular $e_1 \circ h_1 = e_2 \circ h_2$ such that by the pullback property we get unique $h : E \to \{\overline{z} | \varphi \wedge \psi\}$ such that $h_1 = p_1 \circ h, h_2 = p_2 \circ h$. Note that $e' = e_1 \circ h_1 = e_1 \circ p_1 \circ h = e \circ h$. Let now $k : E \to \{\overline{z} | \varphi \wedge \psi\}^{(M)}$ be such that $e' = e \circ k$. We define

$k_1 = p_1 \circ k, k_2 = p_2 \circ k$. Then we have $e_1 \circ k_1 = e_1 \circ p_1 \circ k = e \circ k = e'$ such that by uniqueness of $h_1$ we get $h_1 = k_1 = p_1 \circ k$. Similarly $h_2 = p_2 \circ k$. We now have by uniqueness of $h$ that $k = h$. This completes the proof. $\triangle$

**Exericse 2.** (3 + 7 points) Let $T$ be a theory and $M$ a model of $T$. Prove the following:

**a.** Let $p(\overline{z}), q(\overline{z})$ be formulas with free variables among $\overline{z} : \overline{Z}$. Then we have

$$\{\overline{z} | p(\overline{z}) \wedge q(\overline{z})\}^{(M)} \leq \{\overline{z} | p(\overline{z})\}^{(M)}.$$

*Solution.* By Lemma 4.1 it suffices to display a deduction of $T, (p(\overline{z}) \wedge q(\overline{z})) \vdash_{\overline{z}} p(\overline{z})$. An example is given below

$$\dfrac{\dfrac{}{T, p(\overline{z}) \wedge q(\overline{z}) \vdash_{\overline{z}} p(\overline{z}) \wedge q(\overline{z})}\,(1.1)}{T, p(\overline{z}) \wedge q(\overline{z}) \vdash_{\overline{z}} p(\overline{z})}\,(2.2)$$

where the labels indicate the deduction rule used. $\triangle$

**b.** Let now $p(\overline{x}, y)$ be a formula with free variables among $\overline{x} : \overline{X}$ and $y : Y$. Let also $q(y), r(y)$ be formulas with as free variables $y$ or none such that the sequent $q(y) \Rightarrow r(y)$ is in $T$. Then we have

$$\{\overline{x} | \exists y(p(\overline{x}, y) \wedge q(y))\}^{(M)} \leq \{\overline{x} | \exists y(p(\overline{x}, y) \wedge r(y))\}^{(M)}$$

*Solution.* By Lemma 4.1 it suffices to display a deduction of $T, \exists y(p(\overline{x}, y) \wedge q(y)) \vdash_{\overline{x}} \exists y(p(\overline{x}, y) \wedge r(y))$. An example is given below. We have left out the "T"s

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists y(p(\overline{x}, y) \wedge q(y)) \vdash_{x} \exists y(p(\overline{x}, y) \wedge q(y))}\,(1.1)}{p(\overline{x}, y) \wedge q(y) \vdash_{\overline{x},y} p(\overline{x}, y) \wedge q(y)}\,(2.3)}{p(x, y) \wedge q(y) \vdash_{\overline{x},y} q(y)}\,(2.2) \qquad \dfrac{}{q(y) \vdash_{\overline{x},y} r(y)}\,(\mathrm{T} + 1.3)}{p(\overline{x}, y) \wedge q(y) \vdash_{\overline{x},y} r(y)}\,(1.2) \qquad \dfrac{\dfrac{\dfrac{}{\exists y(p(\overline{x}, y) \wedge q(y)) \vdash_{\overline{x}} \exists y(p(\overline{x}, y) \wedge q(y))}\,(1.1)}{p(\overline{x}, y) \wedge q(y) \vdash_{\overline{x},y} p(\overline{x}, y) \wedge q(y)}\,(2.3)}{p(\overline{x}, y) \wedge q(y) \vdash_{\overline{x},y} p(\overline{x}, y)}\,(2.2)}{\dfrac{p(\overline{x}, y) \wedge q(y) \vdash_{\overline{x},y} p(\overline{x}, y) \wedge r(y)}{\exists y(p(\overline{x}, y) \wedge q(y)) \vdash_{\overline{x}} \exists y(p(\overline{x}, y) \wedge r(y))}\,(2.3)}\,(2.2)$$

where the labels indicate the deduction rule(s) used. $\triangle$

**Exercise 3.** (Exercise E.4, 5 points) Prove the following statement which was used in the proof of Lemma 5.1: For an arrow $f : X \to Y$ a monomorphism $m$ representing the subobject graph$(f)$ is an equalizer of the two parallel arrows $f \circ \pi_1, \pi_2 : X \times Y \rightrightarrows Y$.

*Solution.* By definition graph$(f)$ is represented by the mono $\langle \mathrm{Id}_X, f \rangle : X \to X \times Y$ so it suffices to show that $\langle \mathrm{Id}_X, f \rangle$ is an equalizer of $f \circ \pi_1, \pi_2$. Note that

$$f \circ \pi_1 \circ \langle \mathrm{Id}_X, f \rangle = f \circ \mathrm{Id}_X = f = \pi_2 \circ \langle \mathrm{Id}_X, f \rangle.$$

We see that $\langle \mathrm{Id}_X, F \rangle$ makes $f \circ \pi_1, \pi_2$ equal. It remains to prove the equalizer property. Let $e : E \to X \times Y$ make $f \circ \pi_1, \pi_2$ equal. Then we have $h := \pi_1 \circ e : E \to X$. Note now that

$$\pi_1 \circ \langle \mathrm{Id}_X, f \rangle \circ h = \mathrm{Id}_X \circ h = h = \pi_1 \circ e$$

and

$$\pi_2 \circ \langle \mathrm{Id}_X, f \rangle \circ h = f \circ h = f \circ \pi_1 \circ e = \pi_2 \circ e.$$

We see $\langle \mathrm{Id}_X, f \rangle \circ h = e$. Let now $k : E \to X$ also be such that $\langle \mathrm{Id}_X, f \rangle \circ k = e$. Then we have

$$k = \mathrm{Id}_X \circ k = \pi_1 \circ \langle \mathrm{Id}_X, f \rangle \circ k = \pi_1 \circ e = h.$$

We see that $\langle \mathrm{Id}_X, f \rangle$ is indeed an equalizer of $f \circ \pi_1, \pi_2$. $\triangle$