Homework exercise 1

In this exercise you will fill in a gap in the proof of theorem 3.1 (see handout). Recall that we fixed a complete Heyting algebra Ω and some Ω-set (X, δ). Let $A \subset \Omega$ and suppose that we are given an arrow $\alpha_a : 1_a \to (X, \delta)$ for each $a \in A$ such that for all $a, a' \in A$ we have

$$\alpha_a \circ e_{a \wedge a'\ a} = \alpha_a \circ e_{a \wedge a'\ a'},$$

where e_{qp} is the unique arrow $1_q \to 1_p$ for $q \leq p$. Set $p = \bigvee A$ and define $\alpha : \{*_p\} \times X \to \Omega$ by $\alpha(*_p, x) = \bigvee_{a \in A} \alpha_a(*_a, x)$. Show that α is an arrow $1_p \to (X, \delta)$.

Hint. In the previous homework we have seen that any complete lattice that satisfies the infinitary distributive law is a complete Heyting algebra. In this exercise you may use, without proof, the converse of that statement: any complete Heyting algebra satisfies the infinitary distributive law. That is, for any $p \in \Omega$ and any subset $A \subset \Omega$ one has:

$$p \wedge \bigvee A = \bigvee_{a \in A} p \wedge a.$$

Scoring. There are four properties to check, the first three are each rewarded with one point, the fourth is rewarded with two points.

Homework exercise 2

Your goal is to prove that the implication subsheaf $A \to B$ defined during the lecture is indeed a sheaf. Recall that given subsheaves $A, B \subset F$ over some fixed complete Heyting algebra Ω, we define the implication to be

$$(A \to B)_p := \{x \in F(p) \mid \forall q \leq p.\, x|_q \in A_q \Rightarrow x|_q \in B_q\}.$$

Let $p \in \Omega$ and let $Q \subset \Omega$ be such that $\bigvee Q = p$. Let $(x_q \in (A \to B)_q)_{q \in Q}$ be such that for every $q, q' \in Q$, $x_q|_{q \wedge q'} = x_{q'}|_{q \wedge q'}$. Note that this is an arbitrary compatible family.

Show that this compatible family has a unique amalgamation. It may be helpful to split the work as follows:

a. (0.5 points) Show there is some unique $x \in F_p$ such that $x|_q = x_q$ for every $q \in Q$. Conclude that if an amalgamation exists, it must be unique.

b. (0.5 points) Show that if $x \in A_p$ then for every $q \in Q$, $x_q \in A_q$.

c. (1 points) Show that if $x \in A_p$ then $x \in B_p$.

d. (2 points) Let $p' \leq p$ and suppose $x|_{p'} \in A_{p'}$. Show that $x|_{p'} \in B_{p'}$.

e. (1 points) Conclude that $(x_q \in (A \to B)_q)_{q \in Q}$ has a unique amalgamation in $(A \to B)_p$.