Seminar on Models of Intuitionism

Hand-in exercise 5

16 March (due 23 March)

Exercise 1. In the exercises below you may use (without proof) that addition, multiplication, bounded sums, bounded products, bounded minimalisation and sign are all primitive recursive. Let $F\colon \mathbb{N}^{k+1} \to \mathbb{N}$ be primitive recursive. Show that the following functions are all primitive recursive:

(a) $\lambda x.x!$ *(0.5 points)*

(b) *(1 point)*

$$\lambda x.\operatorname{pd}(x) \text{ where } \operatorname{pd}(x) = \begin{cases} 0 & \text{if } x = 0 \\ x' & \text{if } x = x' + 1 \end{cases}$$

$$\lambda xy.x - y \text{ where } x - y = \begin{cases} x - y & \text{if } y \leq x \\ 0 & \text{else} \end{cases}$$

$$\lambda xy.x \leq y \text{ where } (x \leq y) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{else} \end{cases}$$

$$\lambda xy.x = y \text{ where } (x = y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{else} \end{cases}$$

(c) *(1 point)*

$$\lambda \vec{x}z.\forall_{y<z}(F(\vec{x}, y) = 0) \text{ where } \forall_{y<z}(F(\vec{x}, y) = 0) = \begin{cases} 0 & \text{if } \forall y < z \text{ we have } F(\vec{x}, y) = 0 \\ 1 & \text{else} \end{cases}$$

$$\lambda xy.x \nmid y \text{ where } x \nmid y = \begin{cases} 0 & \text{if } x \nmid y \\ 1 & \text{else} \end{cases}$$

(d) $\lambda x.\operatorname{prime}(x)$ where $\operatorname{prime}(x) = \begin{cases} 0 & \text{if } x \text{ is prime} \\ 1 & \text{else} \end{cases}$ *(1 point)*

(e) $\lambda n.p_n$ where p_n is the n-th prime number and $p_0 = 1$ *(2 points)*

Exercise 2. A recursive function[1] $F\colon \mathbb{N} \to \mathbb{N}$ is called *self-describing* if $F = \varphi_e$ and e is the least integer k with $F(k) \neq 0$. Prove that there exists a self-describing function. *(1.5 points)*

Exercise 3. Let $H = \{(f, x) \mid \varphi_f(x) \text{ is defined}\}$ be the Halting Problem. Let K be the *Diagonal Halting Problem*, i.e. $K = \{x \mid \varphi_x(x) \text{ is defined}\}$.

(a) Show that there is a recursive function F such that $(x, y) \in H$ iff $F(x, y) \in K$. *(2.5 points)*

(b) Conclude that χ_K is not recursive. *(0.5 points)*

[1] A recursive function is a total partial recursive function.