THE JOURNAL OF SYMBOLIC LOGIC
Volume 10, Number 4, Dec. 1945

ON THE INTERPRETATION OF INTUITIONISTIC NUMBER THEORY

S. C. KLEENE

The purpose of this article is to introduce the notion of "recursive realizability."[1]

1. Let P be some property of natural numbers. Consider the existential statement, "There exists a number n having the property P." To explain the meaning which this has for a constructivist or intuitionist, it has been described as a partial judgement, or incomplete communication of a more specific statement which says that a certain given number n, or the number n obtainable by a certain given method, has the property P.[2] The meaning of the existential statement thus resides in a reference to certain information, which it implies could be stated in detail, though the trouble is not taken to do so. Perhaps the detail is suppressed in order to convey a general view of some fact.

The information to which reference is made should be thought of as possibly comprising other items besides the value of n or method for obtaining it, namely such items as may be necessary to complete the communication that that n has the property P.

Consider next the generality statement, "All numbers n have the property P." The accompanying explanation which has been given for this is that it is a hypothetical assertion about whatever particular n might be given. We now propose, without excluding this motif, likewise to regard the generality statement as an incomplete communication of a more specific statement, namely of one which gives an effective general method for obtaining, to any particular value of n, the information implicit in the assertion that that n has the property P.

As a third example, consider the implication, "A implies B." This we now propose to interpret intuitionistically as an incomplete communication of

Received July 12, 1945. Preliminary report presented to the Association for Symbolic Logic and the American Mathematical Society, December 31, 1941.

[1] The bracketed numbers in the footnotes refer to the bibliography at the end of the article. The theory of recursive functions presupposed in this paper is recapitulated in the author's **[14]** or very briefly in **[15]**. Several of the footnotes accordingly contain, in addition to the source references, references to those papers.

Much of the detailed investigation of the notion of realizability introduced now is the work of David Nelson. In order to complete the present account of the notion, we shall draw upon six fundamental results of his **[16]**, numbered (I)–(VI) below. The main conclusions reached in this article are therefore joint results of Nelson and the author.

(The work was done in the following sequence. Heuristically, the point of departure was the author's **[14]** §16. The notion of realizability was proposed by the author in a seminar at the University of Wisconsin in the spring of 1941. Nelson undertook to prove (I). The author saw this proof in the early summer of 1941. In February 1942, the author completed the first draft of the present paper, in which the other five propositions (II)–(VI) were formulated as conjectures. Nelson undertook the proof of (II)–(VI) as a continuation of his part of the problem. The first draft of Nelson's **[16]**, containing proofs of all six propositions, was completed in the winter of 1944–5.)

[2] **[9]** p. 32.

another statement, that statement to be one which gives an effective general procedure by means of which, whenever information which completes A is supplied, information which completes B can be found.

In this paper, by treating from this standpoint each of the statement forms of the predicate calculus, presupposing some predicates of natural numbers, we shall reach a definition of what items of information would "realize" a given number-theoretic statement constructed from our primitives. The property of "realizability" will then be a kind of intuitionistic truth notion for the number-theoretic statements of the class in question.

2. The analysis which leads to this truth definition is not to be regarded as more than a partial analysis of the intuitionistic meaning of the statements, since it takes over without analysis, or leaves unanalyzed, the component of evidence. Suppose for example that a mathematician makes a discovery about the natural numbers, and states a generality proposition as an incomplete communication of his discovery. A complete communication of the discovery to another person would have to provoke in the latter the same discovery. For this it may not be enough that the second person be given merely a method for reproducing the facts of the discovery as they apply to whatever particular cases of the proposition he may please to examine.

The limitation in the extent of the analysis appears formally in that the truth definition employs the quantifiers and the connectives of the propositional calculus, as do the statements which are the objects of the definition.

When the quantifiers and propositional connectives in the truth definition are interpreted intuitionistically, we can consider that the reader is supplying the reference to items of evidence in his reading of the definition. The definition then pairs, within intuitionistic number theory, to each proposition expressible in our primitives, a necessary and sufficient condition for the same.

If the quantifiers and propositional connectives in the truth definition are interpreted in the sense of classical mathematics, the definition gives a classical necessary condition for the intuitionistic proposition.

This will be made more precise below (§§13ff.) on the basis of results obtained by David Nelson.

3. A key to the truth definition which we shall set up is the thesis of Church which identifies the effective calculability of a function of natural numbers with its general recursiveness in the sense of Herbrand and Gödel, or the Turing description of computing machines, from which the identity of the computable functions with the general recursive functions can be proved.[3] This will enable us to represent effective general methods, to which reference is made in our analysis of generality statements, by general recursive functions.

By availing ourselves of the device of Gödel numbering, as applied in the theory of general recursive functions by the present writer (the numbers used being called Gödel numbers of the recursive functions, or numbers which define the functions recursively), the items of information which realize a statement

[3] [1], [6], [17], [18]; [14] §§2, 12, [15] §1.

will be compressed into a single natural number.[4] A realization number by itself of course conveys no information; but given the form of statement of which it is a realization, we shall be able in the light of our definition to read from it the requisite information.

The operation of implication is particularly a critical one for the interpretation of intuitionistic logic. Here a sufficient analysis of it for our purpose is obtained by use of the author's notion of partial recursive function.[5] Negation is then treated as the implication of an absurd statement.[6]

4. In giving our definition precisely, in the next section, we shall take the statements to which it applies to be the formulas of a symbolic object language. We shall merely sketch the distinctive features of this language, the method by which such a language is set up being well enough known to within inessential variations of detail.

The successive natural numbers shall be represented in the language by the *numerals* 0, 0′, 0″, $\cdots$, abbreviated as 0, 1, 2, $\cdots$. There shall be *variables* a, a_1, a_2, $\cdots$, x, x_1, x_2, $\cdots$, interpreted as ranging over the natural numbers. There shall be *function symbols*, representing certain given general recursive number-theoretic functions, including the successor function ′. There shall be *predicate symbols*, representing certain given general recursive predicates, i.e., propositional functions of natural numbers, including the equality predicate =. The precise selection of functions and predicates thus represented by single symbols we leave open, several possibilities being:

A. Functions: ′, + (sum), · (product). Predicates: =.
B. Functions: ′. Predicates: =, $a+b=c$, $a \cdot b=c$.
C. Functions: all primitive recursive functions. Predicates: =.
D. Functions: ′. Predicates: all primitive recursive predicates.
E. All primitive recursive functions and predicates.

The selection may be intermediate between A or B and E, or it may include some non-primitive general recursive functions and predicates.

The usual syntactical rules shall govern the formation from the foregoing primitives of expression called *terms* which represent constant or variable natural numbers, and expressions called *elementary formulas* which represent constant or variable propositions about natural numbers. From the assigned interpretations of the symbols, and the usual interpretation of the operations of composition (explicit definition), each term $\mathrm{t}(\mathrm{a}_1,\cdots,\mathrm{a}_n)$ containing exactly the n distinct variables $\mathrm{a}_1,\cdots,\mathrm{a}_n$ must represent a known general recursive function $t(a_1,\cdots,a_n)$ of n variables, and each elementary formula $\mathrm{F}(\mathrm{a}_1,\cdots,\mathrm{a}_n)$ containing exactly the n distinct variables $\mathrm{a}_1,\cdots,\mathrm{a}_n$ must represent a known general recursive predicate $F(a_1,\cdots,a_n)$ of n variables. In particular, a term t containing no variables must represent a known natural number t, and an elementary formula F containing no variables must represent a proposition F either known

[4] [11]; [14] §§4, 7, [15] §10.
[5] [12]; [14] §6, [15] §10.
[6] [10] p. 359.

to be true or known to be false. In the case of the truth of this proposition, we shall say simply that F is *true*.

The *formulas* shall be comprised of the elementary formulas, together with the additional formulas which can be formed from them by employing the *logical symbols* of the predicate calculus in accordance with the usual syntactical rules. These logical symbols we specify to be the propositional connectives & (and), $\vee$ (or), $\supset$ (implies), $\neg$ (not), and the quantifiers $\exists$x (there exists an x such that), $\forall$x (for all x). In this case the interpretations shown parenthetically are merely verbal, our problem being to analyze them presupposing the interpretations already given for the terms and elementary formulas.

Two notational conventions will be useful. If "x", "x_1", "y", $\cdots$ represent certain natural numbers intuitively, then "$\mathbf{x}$", "$\mathbf{x}_1$", "$\mathbf{y}$", $\cdots$ shall represent the corresponding numerals, and conversely.

If $\mathrm{a}_1, \cdots, \mathrm{a}_n$ are distinct variables, and if "$\mathrm{E}(\mathrm{a}_1, \cdots, \mathrm{a}_n)$" is explicitly introduced to stand for a certain term or formula, thereafter for any other appropriate set of terms $\mathrm{p}_1, \cdots, \mathrm{p}_n$, "$\mathrm{E}(\mathrm{p}_1, \cdots, \mathrm{p}_n)$" shall stand for the result of substituting $\mathrm{p}_1, \cdots, \mathrm{p}_n$ for the free occurrences of $\mathrm{a}_1, \cdots, \mathrm{a}_n$, respectively, throughout $\mathrm{E}(\mathrm{a}_1, \cdots, \mathrm{a}_n)$. (To find out what $\mathrm{E}(\mathrm{p}_1, \cdots, \mathrm{p}_n)$ stands for, one must always go back to the original expression $\mathrm{E}(\mathrm{a}_1, \cdots, \mathrm{a}_n)$ and the original variables $\mathrm{a}_1, \cdots, \mathrm{a}_n$ with which "E" was introduced.)

(A list of the fonts of type employed technically in the present article with their uses may serve to forestall confusions. (i) Roman letters "A", "x" are used in metamathematical designations for expressions of the symbolic object language or formal system, such as formulas and formal variables. (ii) Italic letters "A", "x" are used to designate intuitive mathematical objects such as propositions, predicates, and natural numbers. This is the same use of them as in ordinary mathematical discourse. In some contexts, a roman letter or letters and the same letter or letters in italics may be correlated to designate respectively, some formal object, and the intuitive object which the formal object represents under the interpretation of the formal system. (iii) A heavy type letter "$\mathbf{x}$" is a metamathematical designation, used always in correlation with the same letter in italics, the latter standing for a natural number which may be variable or constant, and the former standing for the corresponding numeral. (iv) Script letters "$\mathcal{A}$", "$\varkappa$", used only in §10, are particular predicate and individual variables of a formal symbolism for predicate calculus. For purists in the matter of designation, our use of them is autonymous. (v) A shaded letter "$\mathbb{A}$" is a metamathematical designation, used always in correlation with the same letter in roman type, the latter designating a formula, and the former designating another formula correlated to that formula in the manner to be described in §§11 and 12).

5. A natural number will be said to *realize* a formula, or to be a *realization* (or *realization number*) of the formula, under each of the following circumstances, and only those.

I. If $\mathrm{A}(\mathrm{y}_1, \cdots, \mathrm{y}_m)$ is a formula containing exactly the distinct free variables $\mathrm{y}_1, \cdots, \mathrm{y}_m$ in order of first free occurrence, and if e *realizes* $\forall \mathrm{y}_1 \cdots \forall \mathrm{y}_m \mathrm{A}(\mathrm{y}_1, \cdots, \mathrm{y}_m)$, then e also *realizes* $\mathrm{A}(\mathrm{y}_1, \cdots, \mathrm{y}_m)$.

II. The remaining seven clauses govern the assignment of realization numbers to formulas not containing free variables.

1. An elementary formula F without free variables is *realized* by 0 if it is true.

For Clauses 2–5, A and B are formulas without free variables.

2. If a *realizes* A and b *realizes* B, then $2^a \cdot 3^b$ *realizes* A & B.

3. If a *realizes* A, then $2^0 \cdot 3^a$ *realizes* A ∨ B. Also, if b *realizes* B, then $2^1 \cdot 3^b$ *realizes* A ∨ B.

4. The formula A ⊃ B is *realized* by the Gödel number e of a partial recursive function ϕ such that, whenever a *realizes* A, then $\phi(a)$ *realizes* B.

5. If e *realizes* A ⊃ 1 = 0, then e also *realizes* ¬A. This clause is so written that, if we prefer, we can omit ¬ from the undefined symbolism, and take ¬A to be an abbreviation for A ⊃ 1 = 0.

For Clauses 6 and 7, x is a variable, and A(x) is a formula without free variables other than x. The conventions introduced at the end of §4 are used in stating these clauses.

6. If a *realizes* A(***x***), then $2^x \cdot 3^a$ *realizes* ∃xA(x).

7. The formula ∀xA(x) is *realized* by the Gödel number e of a general recursive function ϕ such that, for every x, $\phi(x)$ *realizes* A(***x***).

A formula is said to be *realizable* (or *recursively realizable*), if and only if some natural number realizes it.

(This notion of realizability is obviously capable of various generalizations and modifications, in the first instance by employing some other enumerable class of partially and completely defined number-theoretic functions in place of the recursive.)

6. Consider any formula A. Besides the truth definition which we have just set up for A, in the preceding section, there is the direct one which is given by the usual verbal interpretations of the logical symbols.

This is perhaps sufficient explanation of it, but for comparison with the definition of realizability we shall also give it in full in corresponding manner.[7] The letters used in the respective clauses are subject to the same stipulations.

I. $A(y_1, \cdots, y_m)$ is *true*, if $\forall y_1 \cdots \forall y_m A(y_1, \cdots, y_m)$ is *true*.

II. 1. F is *true*, if *F* (i.e., if *F* is a true proposition in the theory of recursive predicates; cf. §4).

2. A & B is *true*, if A is *true* and B is *true*.

3. A ∨ B is *true*, if A is *true* or B is *true*.

4. A ⊃ B is *true*, if A is *true* implies that B is *true* (i.e., if A is *true* only if B is true).

5. ¬A is *true*, if A is not *true*.

6. ∃xA(x) is *true*, if there exists an x such that A(***x***) is *true*.

7. ∀xA(x) is *true*, if, for all x, A(***x***) is *true*.

Into what inferences we may incorporate the propositions "A is true" and "A is realizable" will depend on how the logical constants are being employed in the syntax language, and we shall distinguish between intuitionistic and classical uses.

---

[7] This method of setting up a truth definition is discussed in **[13]**.

7. The following statements are ready consequences of the definitions of §§5 and 6 under either the intuitionistic or the classical usages, except where we single out the classical.[8]

(a) If A is realizable and contains no occurrences of $\supset$ or $\neg$, then A is true.

(b) If $\neg$A is true, and A contains no occurrences of $\supset$ or $\neg$, then $\neg$A is realizable.

(c) Not both A and $\neg$A can be realizable.

(d) If A is realizable, then $\neg\neg$A is realizable.

(e) If A is unrealizable and contains no free variables, then $\neg$A is realizable.

(f) If A is unrealizable and contains no free variables, then by (e) and (c), $\neg\neg$A is unrealizable.

(g) By the classical law of the excluded middle, either A is realizable or A is unrealizable. In the second case, if A contains no free variables, by (e), $\neg$A is realizable. Thus, classically, if A contains no free variables, either A is realizable or $\neg$A is realizable, despite the fact that the law of the excluded middle is not affirmed within the intuitionistic logic.

(h) From (g), classically, if A contains no free variables, then A $\vee$ $\neg$A is realizable.

(i) From (f) by contraposition and the classical law of double negation, if A contains no free variables and $\neg\neg$A is realizable, then A is realizable, despite the fact that the law of double negation is not affirmed within the intuitionistic logic.

(j) If B is realizable, then A $\supset$ B is realizable.

(k) Generalizing (e), if A is unrealizable and contains no free variables, then A $\supset$ B is realizable.

(l) By the classical law of the excluded middle, either $\neg\neg$A is realizable or $\neg\neg$A is unrealizable. Hence by (i), (j), and (k), classically, if A contains no free variables, then $\neg\neg$A $\supset$ A is realizable.

(m) Using the theorem on the least-number operator from the theory of general recursive functions, if $\exists$xA(x) is true, and A(x) is elementary, then $\exists$xA(x) is realizable.[9] (The result can be extended to the case that A(x) merely contains no occurrences of $\exists$ or $\forall$.)

8. Does our interpretation of the number-theoretic formulas by the notion of realizability satisfy the formal postulates of the intuitionists for number theory?

David Nelson has obtained the following result.[10]

(I) For a formal system which has a class of formulas as described in §4, and which has as deductive postulates those of the intuitionistic predicate calculus with equality (expressed as schemata), the Peano axioms including mathematical induction, and arbitrary additional realizable axioms (e.g., the recursion equations for + and $\cdot$), every formula provable in the system is realizable. In other words, every formula deducible from realizable formulas

[8] For several of the statements, a rigorous proof requires the use of some particular recursive functions, such as may be found in [4] or [6] or [9] with [11] and [12].

[9] [1] IV, [11] V; [14] §3, [15] §10.

[10] [16] Theorem 1.

by means of the intuitionistic predicate calculus with equality and the Peano axioms is realizable.

This also answers another question. Church's thesis that all effectively calculable number-theoretic functions are general recursive arose separately from the intuitionistic formalization of logic. But both correspond to the same constructivist standpoint. Accepting Church's thesis, should we not be able to show that the postulates of the intuitionists do not allow them to prove the existence of any other than a general recursive function?

The existence of a function $y = \psi(x_1, \cdots, x_n)$ is expressed by a formula of the form $\forall x_1 \cdots \forall x_n \exists y B(x_1, \cdots, x_n, y)$ where $B(x_1, \cdots, x_n, y)$ contains besides $x_1, \cdots, x_n, y$ no free variables. Applying (I) to this formula, Nelson states the following corollary.

If $\forall x_1 \cdots \forall x_n \exists y B(x_1, \cdots, x_n, y)$ is provable in the system of (I), then there exists a general recursive function $\psi(x_1, \cdots, x_n)$ such that, for every set $x_1, \cdots, x_n$ of natural numbers, the formula $B(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y})$, where $y = \psi(x_1, \cdots, x_n)$, is realizable.

The author's earlier conjecture in this direction was expressed in terms of the notion of "recursive fulfillability":[11] the formula $\forall x_1 \cdots \forall x_n \exists y B(x_1, \cdots, x_n, y)$ is *recursively fulfillable*, if there exists a general recursive function $\psi(x_1, \cdots, x_n)$ such that, for every set $x_1, \cdots, x_n$ of natural numbers, the formula $B(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y})$, where $y = \psi(x_1, \cdots, x_n)$, is true.

The present results in terms of recursive realizability are of more interest, since the notion of recursive fulfillability is limited to formulas of the special form $\forall x_1 \cdots \forall x_n \exists y B(x_1, \cdots, x_n, y)$, and stops with the interpretation of the prefixed quantifiers $\forall x_1 \cdots \forall x_n \exists y$.

(The original form of the conjecture, namely that if $\forall x_1 \cdots \forall x_n \exists y B(x_1, \cdots, x_n, y)$ is intuitionistically provable, then it is recursively fulfillable, can be discussed on the basis of Nelson's results as follows. Let a modified notion of realizability for a particular formal system of (I) be defined by altering three clauses of the definition in §5 as follows. Clause II 3: After "a realizes A," insert "and A is provable"; and after "b realizes B," insert "and B is provable." Clause II 4: After "a realizes A," insert "and A is provable." Clause II 6: After "a realizes $A(\boldsymbol{x})$," insert "and $A(\boldsymbol{x})$ is provable." Also let the particular formal system have the property that every true elementary formula without free variables is provable. It may be verified that Nelson's proof of (I) and the corollary holds good in this case for the modified notion of realizability. The formula $B(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y})$ of the corollary is then not only realizable but is provable in the formal system. In order to state a result both intuitionistically and classically, let the system of (I) be a subsystem of the classical system. Then $B(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n, \boldsymbol{y})$, since provable, must be true, intuitionistically and classically.)

9. Can a formula be true classically but unrealizable? The author has given informally an example of a proposition which holds classically but is not recursively fulfillable.[12] If this proposition is suitably formalized in the present

[11] [14] §16.

[12] [14] §16, using the example given on p. 71.

symbolism, supposing that the symbolism is adequate, as it would be under each of Plans A–E of §4, we shall obtain a formula of the form $\forall x \exists y B(x,y)$, where B(x,y) is a formula which contains besides x and y no free variables, and which has the property that, for each x and y, $B(\boldsymbol{x},\boldsymbol{y})$ is true if realizable. (The informal proposition involves the negation of a certain primitive recursive predicate T_1, but if we are formalizing under any one of Plans C–E of §4, and treat both T_1 and its negation as simple predicates, then B(x,y) will contain no occurrences of $\supset$ or $\neg$, and will have the property in consequence of (a) of §7. For any other suitable formalization, a proof of the property of B(x,y) could be drawn from the fact of equivalence of that formalization to one of the formalizations without $\supset$ or $\neg$ in a suitable object language containing both.)

The informal proposition holds, and so the formula $\forall x \exists y B(x,y)$ is true, by the classical law of the excluded middle. This implies that there exists a number-theoretic function ψ such that, for all x, $B(\boldsymbol{x},\boldsymbol{y})$, where $y = \psi(x)$, is true. In the example, the function ψ is in fact unique, and is known to be non-recursive.

Now suppose $\forall x \exists y B(x,y)$ were realizable. By Clauses II 6 and 7 of the definition of realizability, there would have to be a general recursive function ψ such that, for all x, $B(\boldsymbol{x},\boldsymbol{y})$, where $y = \psi(x)$, is realizable, and hence true. Since this cannot be, $\forall x \exists y B(x,y)$ is unrealizable. Note that $\forall x \exists y B(x,y)$ contains no free variables.

Can a formula be realizable, but untrue classically? If A is any formula containing no free variables which is true classically but unrealizable, such as the formula just considered, then by (e), $\neg A$ (or by (k), $A \supset B$ where B is any false formula) is realizable but untrue classically.

10. It is of interest to apply Nelson's result (I) to questions concerning provability in the pure intuitionistic predicate calculus as expressed in terms of proposition and predicate variables $\mathfrak{A}$, $\mathfrak{A}(*)$, $\cdots$. If a formula of the intuitionistic predicate calculus is provable, then it must have the property that every number-theoretic formula in the sense of this paper which comes from it by a substitution of number-theoretic formulas for its proposition and predicate variables is realizable.

We may combine this with the author's result (§9) that a certain formula $\forall x \exists y B(x,y)$ containing no free variables is true classically but unrealizable. The classical demonstration of its truth can be formalized as a deduction by means of the intuitionistic predicate calculus with equality from a certain number-theoretic formula of the form $\forall x(A(x) \vee \neg A(x))$ containing no free variables.

By Nelson's result, therefore, the formula $\forall x(A(x) \vee \neg A(x))$ must itself be unrealizable. Since no free variables are present, by (f), $\neg\neg\forall x(A(x) \vee \neg A(x))$ is also unrealizable (this is for a certain number-theoretic A(x)).

Therefore, by Nelson's result, the formula

$$(1) \qquad \neg\neg\forall x(\mathfrak{A}(x) \vee \neg\mathfrak{A}(x))$$

is unprovable in the intuitionistic predicate calculus.[13]

It has been well known that the simple law of the excluded middle $\mathfrak{A} \vee \neg\mathfrak{A}$ is unprovable, yet its double absurdity $\neg\neg(\mathfrak{A} \vee \neg\mathfrak{A})$ is provable, in the intuitionistic logic.[14] The explanation of this under the interpretation provided by the notion of realizability now appears in that the refutation of $\mathfrak{A} \vee \neg\mathfrak{A}$ requires the presence of a free individual variable representing generality. To complete the picture, the double absurdity is unprovable when this individual variable is present and is universally quantified so that the double absurdity applies to the generality statement.

This result naturally entails the unprovability of various other formulas in the intuitionistic predicate calculus, of which we shall mention several. We know that $\forall x\neg\neg(\mathfrak{A}(x) \vee \neg\mathfrak{A}(x))$ is provable. Were also

$$(2) \qquad \forall x\neg\neg\mathfrak{A}(x) \supset \neg\neg\forall x\mathfrak{A}(x)$$

provable, we could thence deduce (1). Therefore (2) is unprovable.

We know that $\forall x\neg\neg\mathfrak{A}(x) \supset \neg\exists x\neg\mathfrak{A}(x)$ is provable.[15] Were also

$$(3) \qquad \neg\forall x\mathfrak{A}(x) \supset \exists x\neg\mathfrak{A}(x)$$

provable, applying contraposition to the latter, we could deduce (2). Therefore (3) is unprovable.

No substitution with free individual variables having been used in these deductions, it follows that the absurdity of the absurdity of each of (2) and (3) are also unprovable.

11. Let A be a formula, let $y_1, \cdots, y_m$ ($m \geqq 0$) be the free variables of the formula in the order of their first free occurrences, and let the formula also be denoted by "$A(y_1, \cdots, y_m)$."

We have thus far been considering the formula as a formal expression for a proposition, with its free variables expressing generality. But the formula also serves as a formal expression for a predicate, i.e., propositional function, with its free variables in the rôle of the parameters or independent variables. We shall now distinguish between the interpretations of the formula by a proposition and the interpretations by a predicate.

Let us catalog the several interpretations.

Under the direct interpretation of A by a proposition, this proposition is *the proposition* "A is true" as used above, which is synonymous with "$\forall y_1 \cdots \forall y_m A(y_1, \cdots, y_m)$ is true" and with "for all $y_1, \cdots, y_m$, $A(y_1, \cdots, y_m)$ is true."

[13] In the summer of 1941, after Nelson and the author had this result, Gödel communicated to the author in conversation that he had established the same fact, at about the same time as they, by a different method also using partial recursive functions.

[14] [7] p. 52 and p. 56.

[15] [8] p. 18.

Under the direct interpretation of A by a predicate, we shall speak of this predicate as *the predicate* "A is true"; it is "$A(y_1,\cdots,y_m)$ is true" considered as predicate of $y_1,\cdots,y_m$.

Under the realizability interpretation of A by a proposition, this proposition is *the proposition* "A is realizable" as used above, which is synonymous with "$\forall y_1\cdots\forall y_m A(y_1,\cdots,y_m)$ is realizable." But we refrain from identifying it with "for all $y_1,\cdots,y_m$, $A(y_1,\cdots,y_m)$ is realizable." Classically, the formula $\exists yB(x,y)$ from §9 is a counter example, since $\forall x\exists yB(x,y)$ is unrealizable, but in fact, using (h), classically, for every x, $\exists yB(x,y)$ is realizable.

Under the realizability interpretation of A by a predicate, we shall speak of this predicate as *the predicate* "A is realizable"; it is "$A(y_1,\cdots,y_m)$ is realizable" considered as predicate of $y_1,\cdots,y_m$.

In the absence of the qualifying word "proposition" or "predicate," we shall understand as above the interpretations by a proposition.

When there are no free variables (i.e., $m=0$), the predicate interpretations coincide with the proposition interpretations.

We now ask whether, to each particular formula A, the realizability predicate for A can be expressed directly in the formal symbolism. More precisely, to each A, can we find another formula $\mathbb{A}$ such that the predicates "A is realizable" and "$\mathbb{A}$ is true" are equivalent? In the next section we shall answer this question in the affirmative, under the supposition that the object language has adequate means for the expression of certain primitive recursive predicates.

When we have this result for predicates, we can at once answer the corresponding question for propositions. Since "$\forall y_1\cdots\forall y_m A(y_1,\cdots,y_m)$ is realizable" and "for all $y_1,\cdots,y_m$, $A(y_1,\cdots,y_m)$ is realizable" are not in general identified, we cannot in general use the same formula $\mathbb{A}$ as we use when A is interpreted by a predicate. However let B be the closure $\forall y_1\cdots\forall y_m A(y_1,\cdots,y_m)$ of A. Then the propositions "A is realizable" and "B is realizable" are equivalent; and by the stated result, since B contains no free variables, the propositions "B is realizable" and "$\mathbb{B}$ is true" are equivalent. Thus the propositions "A is realizable" and "$\mathbb{B}$ is true" are equivalent; i.e., $\mathbb{B}$ is the desired formula. If A has no free variables, $\mathbb{A}$ and $\mathbb{B}$ are the same formula.

(By §5 I and II 7, "$\mathbb{B}$ is true" always implies "$\mathbb{A}$ is true"; but, at least on classical grounds, not conversely, in view of the counter example $\exists yB(x,y)$ given above.)

(We have discussed only the case that all of the free variables express generality, and the case that all are predicate variables. In intermediate cases, evidently closures would be taken with respect to just those of the free variables which are intended to express generality.)

12. Let us examine how the predicate "A is realizable" is determined from our definition, for a particular formula A.

In doing so, it will be convenient to use an informal logical symbolism which has been employed in the theory of recursive functions. In this symbolism, "for all x" is expressed by "(x)"; "there exists an x such that" by "(Ex)";

"implies" by "→"; and "and" and "or" by the same symbols "&" and "v", respectively, as in the formal symbolism.

Using this symbolism and a certain primitive recursive predicate T_1 and function U which have a rôle in the theory of recursive functions, Clauses II 4 and 7 of §5 can be restated thus.[16]

II 4. e realizes A ⊃ B, if $(a)[\{a \text{ realizes A}\} \rightarrow (Ey)[T_1(e,a,y)\ \&\ \{U(y) \text{ realizes B}\}]]$.

II 7. e realizes ∀xA(x), if $(x)(Ey)[T_1(e,x,y)\ \&\ \{U(y) \text{ realizes A}(\boldsymbol{x})\}]$.

To obtain the predicate "A is realizable" for the particular A, we start out with "$(Ee)\{e$ realizes $\mathrm{A}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_m)\}$", considering $y_1,\cdots,y_m$ as parameters, and break the realization predicate "e realizes $\mathrm{A}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_m)$" down by applications of Clauses II 2–7. In the process, $\mathrm{A}(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_m)$ is progressively decomposed into parts, and its variables are progressively replaced by numerals under applications of Clauses II 6 and 7.

Using II 4 and 7 in their new forms, at each stage we shall evidently gain an expression for the realizability predicate in terms of certain recursive predicates and functions, the predicate calculus with number variables only, and the realization predicate for the parts.

Eventually, we come to apply II 1 to elementary formulas. Each such elementary formula F will have the form $\mathrm{F}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)$, where $\mathrm{x}_1,\cdots,\mathrm{x}_n$ are the variables which occur in the corresponding part of the original formula A, where $\mathrm{F}(\mathrm{x}_1,\cdots,\mathrm{x}_n)$ is this corresponding part, where $x_1,\cdots,x_n$ are natural numbers, and $\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n$ are the corresponding numerals. We shall be applying II 1 to F, i.e., to $\mathrm{F}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)$, considering $x_1,\cdots,x_n$ as parameters; in other words, we shall be requiring "e realizes $\mathrm{F}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)$" as predicate of $e,x_1,\cdots,x_n$. Now this predicate is precisely "$e=0\ \&\ F(x_1,\cdots,x_n)$" where $F(x_1,\cdots,x_n)$ is the recursive predicate represented by the elementary formula $\mathrm{F}(\mathrm{x}_1,\cdots,\mathrm{x}_n)$ (cf. §4).

Thus, for the particular A, we shall finally reach an explicit expression for the predicate "A is realizable" in terms of certain recursive predicates and functions, numerals, and the predicate calculus with number variables only. These recursive predicates and functions are $2^x \cdot 3^y$, $T_1(e,x,y)$, and $U(y)$, and otherwise only predicates known to be expressible in the formal symbolism. Therefore, if the three mentioned are so expressible, or if at least T_1 and the predicates into the composition of which the functions enter are so expressible (not necessarily by elementary formulas), then the expression can be translated into the formal symbolism. This will be the case under any one of Plans A–E of §4. We thus obtain heuristically the formula $\mathbb{A}$ for the given A.

In conclusion we shall give the definition of $\mathbb{A}$ from A metamathematically, for the case that the requisite functions and predicates are represented in the formal symbolism immediately by like designated terms and formulas. The letters "F," "A," "B," "x," "A(x)" are used subject to the same stipulations as in II 1–7 of §5, except that now free variables are not excluded. The vari-

[16] [14] §§4, 7, [15] §10.

ables introduced below are to be so chosen as to avoid collision with such free variables.

First we define by a metamathematical recursion a formula $e \circledR A$ depending on A, which represents the realization predicate "*e* realizes A."

1. $e \circledR F$ is $e=0 \ \& \ F$.
2. $e \circledR A \,\&\, B$ is $\exists a \exists b[e=2^a \cdot 3^b \ \&\ a \circledR A \ \&\ b \circledR B]$.
3. $e \circledR A \vee B$ is $\exists a[e=2^0 \cdot 3^a \ \&\ a \circledR A] \vee \exists b[e=2^1 \cdot 3^b \ \&\ b \circledR B]$.
4. $e \circledR A \supset B$ is $\forall a[a \circledR A \supset \exists y[T_1(e,a,y) \ \&\ U(y) \circledR B]]$.
5. $e \circledR \neg A$ is $e \circledR A \supset 1=0$.
6. $e \circledR \exists x A(x)$ is $\exists x \exists a[e=2^x \cdot 3^a \ \&\ a \circledR A(x)]$.
7. $e \circledR \forall x A(x)$ is $\forall x \exists y[T_1(e,x,y) \ \&\ U(y) \circledR A(x)]$.

Then we define $\mathbb{A}$, representing the realizability predicate, to be $\exists e[e \circledR A]$. Another notation for $\mathbb{A}$ would be "$\circledR A$". This would be available in complicated work where it might not be convenient to represent each formula considered by a simple capital letter.

Note that both these "$\circledR$" notations are irregular, since except in the elementary case the formulas abbreviated as "$e \circledR A$" and "$\circledR A$" do not contain the formula A directly as part.

13. On the basis of the general considerations adduced at the beginning of this paper, it seems reasonable to affirm the following three statements.

Intuitionistically, if A is true, then A is realizable.

Intuitionistically, if A is realizable, then A is true.

If A is true intuitionistically, then intuitionistically and classically, A is realizable.

Consider any known formal system S for intuitionistic number theory. For the moment, let us confine our attention to formulas A which contain no free variables.

We certainly cannot hope to formalize the first statement by a metamathematical proof that, for every A, $A \supset \mathbb{A}$ is provable in S. For all such intuitionistic systems are subsystems of classical ones. By §9, there is an A such that classically, A is true, but A is unrealizable, i.e., $\mathbb{A}$ is untrue. Therefore for this A, $A \supset \mathbb{A}$ is untrue classically, hence unprovable in the classical system and hence in the intuitionistic, if the classical system is consistent with its interpretation.

Likewise, using the second result of §9, we cannot expect that, corresponding formally to the second statement, for every A, $\mathbb{A} \supset A$ is provable in S.

Rather, the definition of realizability is thought to make explicit certain necessary and intuitionistically sufficient conditions that a proposition hold from the standpoint of the intuitionists, which hitherto have not been made explicit in such formal postulates as the intuitionists have stated. By annexing $A \supset \mathbb{A}$ and $\mathbb{A} \supset A$ to S as new axiom schemata, we should obtain a strengthened formalization S' of the intuitionistic standpoint for number theory, in which the theory would diverge from the classical.

If we now remove the restriction that A contain no free variables, but retain $A \supset \mathbb{A}$ and $\mathbb{A} \supset A$ as the form of the new schemata, we shall be formalizing the conditions expressed by the first two statements, not only for propositions but for predicates as well.

Nelson's result (I) that for a suitable S, if A is provable, then A is realizable, constitutes a formalization of the third statement. Nelson's proof is in terms of a classical or an unstrengthened intuitionistic syntax language sufficiently strong otherwise to contain the general concept of realizability for the formulas of S.

Nelson's results can be used to secure the consistency of the strengthened intuitionistic number theory by the method of interpretation. We use his aforementioned result (I) in the form that every formula deducible in S from realizable formulas is realizable. Nelson has obtained a second result.[17]

(II) For any formula A, the formulas A $\supset$ $\mathbb{A}$ and $\mathbb{A}$ $\supset$ A are realizable.

From (I) and (II) it follows that in the enlarged system S' only realizable formulas are provable. If each formula A is interpreted as meaning "A is realizable," or in other words read as though it were $\mathbb{B}$ where B is the closure of A, then only correct formulas are provable in S'.

14. The present discussion is built on the author's Gödel numbering of the recursive functions in terms of his predicate T_1 and function U. A straightforward formalization of this numbering has appeared to involve rather unmanageable complexities. Nelson has circumvented this difficulty, and has formalized his proofs of realizability ((I) and (II)) as they apply to particular formulas (as well as to particular classes of formulas of the form $A(y_1,\cdots,y_m)$ where $y_1,\cdots,y_m$ range over all natural numbers) to obtain results which can be summarized as follows, for a suitable S which Nelson has described.[18]

(III) If A is deducible in S from $A_1,\cdots,A_k$ ($k\geqq 0$), and if $B,B_1,\cdots,B_k$ are the respective closures of $A,A_1,\cdots,A_k$, then $\mathbb{B}$ is deducible in S from $\mathbb{B}_1,\cdots,\mathbb{B}_k$. (Likewise, if A is deducible in S from $A_1,\cdots,A_k$, and if $B,B_1,\cdots,B_k$ are the respective closures of $A,A_1,\cdots,A_k$ with respect to the free variables of $A_1,\cdots,A_k$ not held constant in the deduction, then $\mathbb{B}$ is deducible in S from $\mathbb{B}_1,\cdots,\mathbb{B}_k$.)

(IV) For each formula A, if C and D be the respective closures of A $\supset$ $\mathbb{A}$ and $\mathbb{A}$ $\supset$ A, then $\mathbb{C}$ and $\mathbb{D}$ are provable in S.

These two results of Nelson's give the following elementary metamathematical refinement of the foregoing interpretative consistency proof for the strengthening of intuitionistic number theory (end of §13). In this metamathematical refinement, the use of the general concept of realizability for an arbitrary formula A is eliminated.

Combining the two facts, they relate provability in S' with provability in S, thus:

Let A be any formula, and B its closure. If A is provable in S', then $\mathbb{B}$ is provable in S.

Now take as A the false elementary formula $1=0$. Either system is simply

[17] [16] Theorem 2 Corollary.

[18] [16]. The system S is Nelson's system S_3 introduced in Lemma 15. The results (III) and (IV) appear as Theorem 4 Corollary 4.1 and Theorem 5 Corollary Formulas (iii) and (iv), respectively, each taken in conjunction with Theorem 3.

consistent, if and only if this formula A is unprovable in the system. If it is unprovable in S, then so is it in S', because $\mathbb{B}$ is $\exists e[e=0 \& A]$, from which A is deducible.

The simple consistency of S' is thus reduced to that of S.[19]

(This also secures the consistency of the extension of intuitionistic number theory which was proposed earlier using the notion of recursive fulfillability.[20] When the propositions "$(x)(Ey)A(x,y) \rightarrow$ {for some general recursive ϕ, $(x)A(x,\phi(x))$}" are expressed as formulas in S, using T_1 and U, the formulas expressing the realizability of those formulas will be provable in S, by Nelson's methods, and hence the formulas themselves will belong to S'.)

(If B is the closure of a formula A, then $\mathbb{B} \supset \mathbb{A}$ is provable in S. Therefore (IV) and the provability relation following it hold also without taking closures.)

15. Nelson has also formalized the author's proofs of unrealizability (§7(f) and §9) as follows.[21]

(V) For each formula A, if C is the formula $\neg\neg A$, then $\neg\mathbb{A} \supset \neg\mathbb{C}$ is provable in S.

(VI) If A is the formula $\forall x \exists y B(x,y)$ of §9, then $\neg\mathbb{A}$ is provable in S.

The second of these results of Nelson makes it possible as follows to refine metamathematically the reasoning in §13 that $A \supset \mathbb{A}$ and $\mathbb{A} \supset A$ cannot be provable in S, and that their addition to S gives a system S' which diverges from the classical.

Let S_c be the classical system which results from S by annexing the law of the excluded middle as an axiom schema.

For the first incompleteness result for S, namely that $A \supset \mathbb{A}$ is unprovable in it for a certain A, let A be the formula $\forall x \exists y B(x,y)$ of §9. This formula is actually provable in S_c. Suppose $A \supset \mathbb{A}$ were provable in S, and therefore in S_c which contains S. Then $\mathbb{A}$ would be provable in S_c. But according to (VI), $\neg\mathbb{A}$ is provable in S, and hence in S_c. Thus S_c would be simply inconsistent.

Therefore $A \supset \mathbb{A}$ is unprovable in S, if S_c is simply consistent. By a result of Gödel's,[22] S_c is simply consistent, if S is simply consistent.

For the other incompleteness result (stating it now with the letter "C" instead of "A"), let C be the formula $\neg A$ of the present context. By (VI), $\neg\mathbb{A}$ is provable in S, and hence in S'. Also $A \supset \mathbb{A}$ is an axiom of S'. By contraposition, $\neg\mathbb{A} \supset \neg A$ is provable in S'. Therefore $\neg A$ is provable in S'.

Therefore, by the relation between provability in S' and provability in S mentioned in §14 (applied with $\neg A$ as the "A" of the statement in §14, and with $\mathbb{C}$ as the "$\mathbb{B}$"), $\mathbb{C}$ is provable in S.

---

[19] Gentzen's method of proving consistency, [2], [3], [10] pp. 360 ff., applies to S. As is remarked in [10] p. 334 and p. 368, both the Gentzen proof and the proof by a direct truth definition escape the Gödel limitation [4] by using a predicate, the explicit definition of which would require an enumerably infinite number of quantifiers. The same is true of the consistency proof by the realizability interpretation (§7 (c) and §8).

[20] [14] §16.

[21] [16] Theorem 6 (ii) with Theorem 3; and Theorem 7.2 Corollary.

[22] [5].

Now suppose $\mathbb{C} \supset \mathrm{C}$ were provable in S. Then C would be provable in S; that is, $\neg$A would be provable in S, and hence in S_c. But A is provable in S_c.

Therefore $\mathbb{C} \supset \mathrm{C}$ is unprovable in S, if S_c is simply consistent, and hence if S is simply consistent.

As has appeared in the course of the discussion, A is an example of a formula which is provable in S_c, while its negation $\neg$A is provable in S'.

(Similar formalization can be applied to a remark in §11, to show that for a certain formula A, if B is the closure of A, $\mathbb{B}$ is not deducible in S from $\mathbb{A}$. Let A be the formula $\exists y\mathrm{B}(x,y)$ for the B(x,y) of §9. By Nelson's methods, it can be shown that $\mathbb{A}$ is provable in S_c. But the closure B of this A is the "A" of (VI). Thus $\neg\mathbb{B}$ is provable in S. Were $\mathbb{B}$ deducible in S from $\mathbb{A}$, then S_c, which contains S, would be simply inconsistent, and then S itself would be. Since B is deducible from A in S, this also shows that the closure requirement in (III) cannot be further weakened.)

16. Together Nelson's results (V) and (VI) give the following metamathematical refinement of the proof in §10, which was expressed in terms of the interpretation, that (1) and therefore certain other formulas of the classical predicate calculus are unprovable in the intuitionistic predicate calculus.

Let A, B, C be the number-theoretic formulas $\forall x\exists y\mathrm{B}(x,y)$, $\forall x(\mathrm{A}(x) \vee \neg\mathrm{A}(x))$, $\neg\neg\forall x(\mathrm{A}(x) \vee \neg\mathrm{A}(x))$, respectively, of §10.

The formula A is deducible from B in S. Hence by (III), $\mathbb{A}$ is deducible from $\mathbb{B}$ in S. Hence by the deduction theorem, which certainly applies since no free variables are present, $\mathbb{B} \supset \mathbb{A}$ is provable in S, and by contraposition, so is $\neg\mathbb{A} \supset \neg\mathbb{B}$. Also by (VI), $\neg\mathbb{A}$ is provable in S. Therefore $\neg\mathbb{B}$ is provable in S, and by (V) (applied with B as the "A" of (V)), so is $\neg\mathbb{C}$.

Now S is a subsystem of S', so $\neg\mathbb{C}$ is provable in S'. The formula $\mathrm{C} \supset \mathbb{C}$ is an axiom of S', and by contraposition, $\neg\mathbb{C} \supset \neg\mathrm{C}$ is provable in S'. Therefore $\neg$C is provable in S'.

Now were the formula (1) of §10 provable in the intuitionistic predicate calculus, by substitution C would be provable in S'. Then S' would be simply inconsistent. As we saw in §14, this is impossible if S is simply consistent.

Therefore (1) is unprovable in the intuitionistic predicate calculus, if S is simply consistent.

BIBLIOGRAPHY

CHURCH, ALONZO. [1] *An unsolvable problem of elementary number theory*, ***American journal of mathematics***, vol. 58 (1936), pp. 345–363.

GENTZEN, GERHARD. [2] *Die Widerspruchsfreiheit der reinen Zahlentheorie*, ***Mathematische Annalen***, vol. 112 (1936), pp. 493–565. [3] *Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie*, Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series no. 4 (1938), pp. 19–44, Leipzig (Hirzel).

GÖDEL, KURT. [4] *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, ***Monatshefte für Mathematik und Physik***, vol. 38 (1931), pp. 173–198. [5] *Zur intuitionistischen Arithmetik und Zahlentheorie*, ***Ergebnisse eines mathematischen Kolloquiums***, Colloquium 52, June 28, 1932. [6] ***On undecidable propositions of formal mathematical systems***, mimeographed lecture notes, Institute for Advanced Study, 1934.

HEYTING, AREND. [7] *Die formalen Regeln der intuitionistischen Logik*, ***Sitzungsberichte der Preussischen Akademie der Wissenschaften,*** Physikalisch-mathematische Klasse, 1930, pp. 42–56. [8] ***Mathematische Grundlagenforschung. Intuitionismus. Beweistheorie.*** Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3 no. 4 (1934), Berlin (Springer).

HILBERT, DAVID and BERNAYS, PAUL. [9] ***Grundlagen der Mathematik,*** vol. 1 (1934), Berlin (Springer). [10] ***Grundlagen der Mathematik,*** vol. 2 (1939), ibid.

KLEENE, S. C. [11] *General recursive functions of natural numbers*, ***Mathematische Annalen,*** vol. 112 (1936), pp. 727–742. [12] *On notation for ordinal numbers*, this JOURNAL, vol. 3 (1938), pp. 150–155. [13] *On the term 'analytic' in logical syntax*, ***The journal of unified science (Erkenntnis),*** vol. 9 (preprinted for the fifth International Congress for the Unity of Science, Cambridge, Mass., 1939, pp. 189–192). [14] *Recursive predicates and quantifiers*, ***Transactions of the American Mathematical Society,*** vol. 53 (1943), pp. 41–73. [15] *On the forms of the predicates in the theory of constructive ordinals*, ***American journal of mathematics,*** vol. 66 (1944), pp. 41–58. Errata: Page 43, line 12, read "(4)" instead of "(4*)". Page 54, line 4, read "a" instead of "e".

NELSON, DAVID. [16] *Recursive functions and intuitionistic number theory*, forthcoming.

TURING, A. M. [17] *On computable numbers, with an application to the Entscheidungsproblem*, ***Proceedings of the London Mathematical Society,*** ser. 2 vol. 42 (1937), pp. 230–265. [18] *A correction* (to the same), ibid., ser. 2 vol. 43 (1937), pp. 544–546.

THE UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN