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AN ABSTRACT NOTION OF REALIZABILITY FOR WHICH INTUITIONISTIC PREDICATE CALCULUS IS COMPLETE

H. LÄUCHLI

To each formula A of predicate logic and each assignment p of 'proofs' to the atomic parts of A we shall associate a set $p[A]$, the set of 'proofs of A'. The proofs of $A \to B$ are just the functions from $p[A]$ into $p[B]$; the proofs of $\exists vA$ are the pairs $\langle c, x\rangle$ such that x is a proof of A_c^v (substitution of c for v).

Instead of 'proofs of A' we could as well say 'realizing functionals for A'. In contrast to Kleene's (second version of the) notion of realizability [3], we consider arbitrary, not necessarily countable functionals.

We shall show that A is derivable in Heyting's predicate calculus if and only if there is an explicitly definable functional Θ such that $\Theta \in p[A]$ for all p, i.e. if and only if there is a well defined 'proof' of A which does not make use of the internal structure of proofs of the atomic parts of A.

The 'only if' part will be clear from the known results about realizability. For the proof of the 'if' part we make use of the following analogy between Kripke's semantics for intuitionistic logic [4] and the theory of permutation groups. In the former, we can assert the implication $A \to B$ in a situation H iff in any later situation H' where we can assert A, we also can assert B. In the latter, the following is true: Given sets A, B and a group H of permutations of the elements of $A \cup B$ leaving A and B invariant. Then there is an H-invariant function from A into B iff any subgroup H' with a fixed element in A also has a fixed element in B.

The theorem will be established classically. The corresponding result for propositional logic was announced in an abstract [5].

Similar interpretations have been considered by Dana Scott (derived from Gödel's Dialectica interpretation) and by Goodman, Kreisel, Troelstra, Scott (derived from the intuitionistic notion of 'construction'; see [1]). To my knowledge, completeness of intuitionistic predicate calculus has not been established for any of these interpretations.

1. We consider formulas containing n-place predicate letters, a propositional constant f ('false'), individual constants, variables $u, v, w, \ldots$, connectives $\wedge, \vee, \rightarrow, \exists, \forall$. We write $\neg A$ for $A \rightarrow f$. $F(\Gamma)$ denotes the set of all *closed* formulas with individual constants from a set Γ.

In the following Γ and Π are countably infinite sets, $c_0 \in \Gamma$ is a designated element of Γ, $X \times Y$ denotes the Cartesian product of the sets X and Y, $X \dot{\cup} Y$ is the disjoint union $(\{0\} \times X) \cup (\{1\} \times Y)$, Y^X is the set of all functions from X into Y.

To each formula A, not necessarily closed, we associate a set $S(A)$, the set of 'possible proofs of A':

$S(A) = \Pi$ if A is atomic,
$S(A \wedge B) = S(A) \times S(B)$,
$S(A \vee B) = S(A) \dot{\cup} S(B)$,
$S(A \rightarrow B) = S(B)^{S(A)}$,
$S(\forall v A) = S(A)^{\Gamma}$,
$S(\exists v A) = \Gamma \times S(A)$.

Note that $S(A^v_c) = S(A)$ for all individual constants c. Thus $S(\forall v A)$ can be interpreted as the set of all choice functions which assign to each $c \in \Gamma$ an element of $S(A^v_c)$.

A *proof assignment* is any function p which assigns to every (closed) formula $A \in F(\Gamma)$ a set $p[A]$ such that

$p[f] \subseteq p[A] \subseteq \Pi$ if A is atomic,
$p[A \wedge B] = p[A] \times p[B]$,
$p[A \vee B] = p[A] \dot{\cup} p[B]$,
$p[A \rightarrow B] = \{x \in S(A \rightarrow B)\colon xy \in p[B]$ for all $y \in p[A]\}$,
$p[\forall v A] = \{x \in S(\forall v A)\colon xc \in p[A^v_c]$ for all $c \in \Gamma\}$,
$p[\exists v A] = \{\langle c, x \rangle\colon c \in \Gamma$ and $x \in p[A^v_c]\}$.

Note that $p[A] \subseteq S(A)$ for all p and A. The elements of $p[A \rightarrow B]$ are functions with domain $S(A)$. Thus the identity function on $S(A)$ belongs to $p[A \rightarrow A]$ for each p.

2. Let $\mathscr{D}$ be the least class containing the sets $\{0, 1\}$, Γ, Π, such that whenever $D_1, D_2 \in \mathscr{D}$, then $D_1 \times D_2$, $D_1 \cup D_2$, $D_1^{D_2} \in \mathscr{D}$. The elements of $D_1 \times D_2$ are viewed as functions with domain $\{0, 1\}$. Let $\mathscr{F} = \bigcup \mathscr{D}$. The elements of $\mathscr{F}$ will be called *functionals*. *Simple* functionals are those which can be defined explicitly. The following kind of explicit definition will do: We consider terms built from constants $0, 1, c_0$ and variables, using

the following formation rules: If t, s are terms and x is a variable and $D \in \mathscr{D}$, then $t(s)$, $\langle t, s \rangle$, $\lambda^D x(t)$ are terms. Terms are interpreted as follows:

Let V be an assignment of functionals to variables. Then $V[t]$ is the following functional:

$V[0] = 0$;
$V[1] = 1$;
$V[c_0] = c_0$;
$V[x]$ is the functional assigned to x;
$V[t(s)]$ is the value of $V[t]$ at $V[s]$ if $V[t]$ is a function and $V[s]$ belongs to its domain, $V[t(s)] = 0$ otherwise;
$V[\langle t, s \rangle]$ is the function with domain $\{0, 1\}$ and values $V[t]$ and $V[s]$ at 0 and 1 respectively;
$V[\lambda^D x(t)]$ is the function with domain D, taking the value $V^x_a[t]$ for $a \in D$; V^x_a assigns a to x and agrees with V otherwise.

If t is closed (all variables bound by λ), then the functional $V[t]$ does not depend on V. Simple functionals are by definition those given by closed terms.

EXAMPLE 1. Let A, B, C be closed formulas. Let

$$D = S((A \rightarrow C) \wedge (B \rightarrow C)),$$
$$E = S(A \vee B).$$

Then the term

$$\lambda^D x(\lambda^E y(\langle x0(y1), x1(y1) \rangle (y0)))$$

defines a simple functional which belongs to

$$p[(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)]$$

for all proof assignments p.

EXAMPLE 2. Let

$$D = S(\neg \forall v(R(v) \vee \neg R(v))).$$

Then

$$\lambda^D x(x(\lambda^{\Gamma} x(\langle 1, \lambda^{\Pi} x(x) \rangle)))$$

defines a simple functional which belongs to

$$S(\neg\neg \forall v(R(v) \vee \neg R(v))).$$

3. Let σ be a permutation on $\Gamma \cup \Pi$ (i.e. a one–one function from $\Gamma \cup \Pi$ onto itself), which leaves invariant the sets Γ, Π and the designated element c_0. σ extends in a natural way to a permutation on $\mathscr{F}$: $\sigma 0 = 0$, $\sigma 1 = 1$; if

g is a function then $(\sigma g)x = \sigma(g(\sigma^{-1}x))$. In particular $\sigma\langle a, b\rangle = \langle \sigma a, \sigma b\rangle$.

A functional $\Theta \in \mathscr{F}$ is said to be *invariant*, if $\sigma\Theta = \Theta$ for all such σ. Simple functionals are invariant since c_0, 0, 1 and all sets $D \in \mathscr{D}$ are invariant. On the other hand, there are uncountably many invariant functionals in $\mathscr{F}$ that are not simple.

4. In the following A denotes a closed formula containing no individual constants other than c_0. The symbol $\vdash$ denotes derivability in the intuitionistic predicate calculus.

THEOREM. (1). *If $\vdash A$, then there is a simple functional Θ such that $\Theta \in p[A]$ for all proof assignments p.*

(2). *If not $\vdash A$, then there is p such that $p[A]$ contains no invariant functional.*

COROLLARY. *The following are equivalent (classically):*

(a). $\vdash A$;
(b) $\exists$ *simple* $\Theta, \forall p, \quad \Theta \in p[A]$;
(c). $\forall p \exists$ *simple* $\Theta, \quad \Theta \in p[A]$;
(d) $\exists$ *invariant* $\Theta, \forall p, \Theta \in p[A]$;
(e). $\forall p \exists$ *invariant* $\Theta, \Theta \in p[A]$.

The corollary indicates a certain stability of the property '$\vdash A$'. For conditions (b), (c), (d) and (e), which are quite different intensionally, turn out to coincide with '$\vdash A$' extensionally.

If we drop the restrictions put on Θ, then we get classical logic in one case and an intermediate thing in the other: $\forall p \exists$(*arbitrary*) Θ *with* $\Theta \in p[A]$ *iff A is derivable in classical predicate calculus.* On the other hand, the condition $\exists\Theta\forall p$ holds for some intuitionistically invalid formulas, e.g. for

$$\forall v(R(v) \vee Q) \to (\forall v R(v) \vee Q),$$

but not for all classically valid ones, e.g. not for $Q \vee \neg Q$.

Part (2) of the theorem is not true if only those p are considered with $p[f] = \Lambda$ (empty). The formula $\neg\neg \forall v(R(v) \vee \neg R(v))$, call it $\neg\neg B$, gives a counterexample. For if $p[f] = \Lambda$ then $p[\neg B] = \Lambda$ since $p[B] \neq \Lambda$, B being classically valid. Hence $p[\neg\neg B] = S(\neg\neg B)$. But $S(\neg\neg B)$ contains invariant functionals, as was shown in example 2.

The proof of part (1) of the theorem is a routine variation on the proof of theorem 62, [2] p. 504. The remainder of this paper is devoted to the proof of part (2).

5. Let N be the set of the natural numbers (including 0). Let Σ be the set of all finite sequences of natural numbers (including the empty sequence Λ) together with an 'ideal element' U. Let R be the binary relation on Σ such that sRs' iff either s' is U or s is a (not necessarily proper) initial segment of s'. Let Ψ be a function with domain Σ and countable sets as values, such that whenever sRs' and $s \neq s'$, then $\Psi(s) \subseteq \Psi(s')$ and the complement $\Psi(s') - \Psi(s)$ is infinite. We also assume $\Psi(\Lambda)$ infinite. $F(\Psi(s))$ denotes the class of all closed formulas with individual constants from $\Psi(s)$.

Everything introduced so far will remain fixed. In particular, we shall not vary the function Ψ.

A *model* is a binary function $\Phi(A, s)$, where A ranges over $F(\Psi(U))$ and s over Σ, whose range is the set $\{T, F\}$, and which satisfies the following conditions:

(1). if $\Phi(A, s) = T$, then $A \in F(\Psi(s))$;
(2). if $\Phi(A, s) = T$ and sRs', then $\Phi(A, s') = T$;
(3). if $\Phi(f, s) = T$, then $\Phi(A, s) = T$ for all $A \in F(\Psi(s))$;
(4). $\Phi(A \wedge B, s) = T$ iff $A \wedge B \in F(\Psi(s))$, and $\Phi(A, s) = T$ and $\Phi(B, s) = T$;
(5). $\Phi(A \vee B, s) = T$ iff $A \vee B \in F(\Psi(s))$, and $\Phi(A, s) = T$ or $\Phi(B, s) = T$;
(6). $\Phi(A \to B) = T$ iff $A \to B \in F(\Psi(s))$ and for all s' with sRs', if $\Phi(A, s') = T$ then $\Phi(B, s') = T$;
(7). $\Phi(\forall v A, s) = T$ iff $\Phi(A^v_c, s') = T$ for all s' with sRs' and all $c \in \Psi(s')$
(8). $\Phi(\exists v A, s) = T$ iff $\Phi(A^v_c, s) = T$ for some $c \in \Psi(s)$.

LEMMA 1. *Let $A \in F(\Psi(\Lambda))$ with not $\vdash A$. Then (and only then) there is a model Φ such that $\Phi(A, \Lambda) = F$.*

The proof is clear from Kripke's work [4].

The element U is no bother: Any Φ which is defined on $\Sigma - \{U\}$ can be extended to Σ by setting $\Phi(A, U) = T$ for all $A \in F(\Psi(U))$.

6. In this section we establish a relationship between models Φ and proof assignments p.

Let q be a one-one function from Σ into the set of positive prime numbers. The function φ from Σ into N is defined by $\varphi(\Lambda) = 1$, $\varphi(s{*}n) = \varphi(s) \cdot q(s{*}n)$, $\varphi(U) = 0$ ($s{*}n$ denotes adjunction of the last term n to the sequence s). Let $|$ denote the relation of divisibility. Each non-empty subset of Σ has a greatest lower bound (glb) with respect to R. For $n \in N$ let

$s_n = \text{glb}\{s: n|\varphi(s)\}$ (the set is non-empty since $n|\varphi(U)$). For instance $s_1 = \Lambda$ and $s_0 = s_4 = U$.

The following are simple consequences of the definitions:

(1). $\varphi(s)|\varphi(s')$ iff $s\text{R}s'$,

(2). $n|\varphi(s_n)$,

(3). $n|\varphi(s)$ iff $s_n\text{R}s$,

(4). $n|m$ implies $s_n\text{R}s_m$,

(5). $s_{\varphi(s)} = s$,

Let I denote the set of all integers, including the negative ones. Let n' denote the set of residue classes of I modulo n: $1'$ is a one element set, $0'$ is I. We have $n' \cap m' = \Lambda$ for $n, m \in N$, $n \neq m$.

We now define $\Gamma = \Pi = \bigcup \{\Psi(s_n) \times n' : n \in N\}$. (More precisely: We impose a certain structure on the given countable sets Γ and Π. The structure on Γ is isomorphic to that on Π. Thus, as a notational convenience, we identify Γ with Π.)

The designated element c_0 of Γ is to belong to $\Psi(s_1) \times 1'$. Let $\Gamma_n = \bigcup \{\Psi(s_k) \times k' : k|n\}$. Thus $\Gamma = \Gamma_0$ and $c_0 \in \Gamma_1$ and

(6). $n|m$ implies $\Gamma_n \subseteq \Gamma_m$.

The elements of Γ are ordered pairs. If $c \in \Gamma$, let c^- denote its first component: $c^- \in \Psi(s_n)$ for some n. In virtue of (4), if $k|n$ then $\Psi(s_k) \subseteq \Psi(s_n)$. Hence

(7). $c \in \Gamma_n$ implies $c^- \in \Psi(s_n)$.

The converse is not true for all $c \in \Gamma$, but we have

(8). if $d \in \Psi(s_n)$ then there is $c \in \Gamma_n$ with $c^- = d$.

If $A \in F(\Gamma)$, let A^- denote the formula obtained by replacing each individual constant c by c^-.

(9). $A \in F(\Gamma_n)$ implies $A^- \in F(\Psi(s_n))$.

Let σ be the following permutation on Γ: If $c = \langle d, i/n \rangle \in \Psi(s_n) \times n'$, then $\sigma c = \langle d, (i+1)/n \rangle$. Then, for all $c \in \Gamma$ and $n \in N$ we have

(10). $\sigma^n c = c$ iff $c \in \Gamma_n$.

In particular $\sigma c_0 = c_0$ since $c_0 \in \Gamma_1$.

To each model Φ we associate a proof assignment p as follows. Let $A \in F(\Gamma)$ be atomic. We define

$$p[A] = \bigcup \{\Psi(s_k) \times k' : \Phi(A^-, s_k) = T \text{ or } \Phi(f, s_k) = T\}.$$

The requirement $p[f] \subseteq p[A] \subseteq \Pi$ for atomic A is satisfied. $p[A]$ only depends on A^-. This carries over to compound formulas. Hence for all $A, B \in F(\Gamma)$

(11). $A^- = B^-$ implies $p[A] = p[B]$.

Furthermore it is easy to see that

(12). $\sigma(p[A]) = p[A]$, for all A in $F(\Gamma)$.

LEMMA 2. *For all $n \in N$ and $A \in F(\Gamma_n)$, σ^n has fixed elements in $p[A]$ if and only if $\Phi(A^-, s_n) = T$.*

(Thus we have a world constant σ such that for all Φ there is a p such that lemma 2 is true.)

PROOF. The proof is by induction on the complexity of A:

1. Let A be atomic. By (10), σ^n has fixed elements in $p[A]$ iff $\Gamma_n \cap p[A] \neq \Lambda$, i.e. iff there is a k, $k|n$, such that $\Phi(A^-, s_k) = T$ or $\Phi(f, s_k) = T$. By (4), the latter holds iff $\Phi(A^-, s_n) = T$ or $\Phi(f, s_n) = T$. Since $A \in F(\Gamma_n)$ (9) gives $A^- \in F(\Psi(s_n))$. Thus $\Phi(f, s_n) = T$ implies $\Phi(A^-, s_n) = T$. Hence σ^n has fixed elements in $p[A]$ iff $\Phi(A^-, s_n) = T$.

2. Let A be $B \wedge C$ or $B \vee C$. The induction step is clear.

3. Let A be $B \rightarrow C$. '*only if*': let $x \in p[A]$ with $\sigma^n x = x$. Consider s with $s_n\text{R}s$ and $\Phi(B^-, s) = T$. Let $m = \varphi(s)$. Then $n|m$ by (3), and hence $B \in F(\Gamma_m)$ by (6). By (5) s is s_m. The induction hypothesis gives an element $y \in p[B]$ with $\sigma^m y = y$. Now $xy \in p[C]$ and $\sigma^m(xy) = (\sigma^m x)(\sigma^m y) = xy$ and $C \in F(\Gamma_m)$. The induction hypothesis gives $\Phi(C^-, s) = T$. Hence $\Phi(A^-, s_n) = T$.

'*if*': Let $\Phi(A^-, s_n) = T$. If $n|m$ then $s_n\text{R}s_m$ (by (4)) and hence $\Phi(B^-, s_m) = T$ implies $\Phi(C^-, s_m) = T$. By the induction hypothesis for each m with $n|m$, if σ^m has fixed elements in $p[B]$, then it also has fixed elements in $p[C]$. Let H be the permutation group generated by σ^n, and let $H(y)$ denote the largest subgroup of H leaving y fixed. Since any subgroup of H is generated by σ^m for some m with $n|m$ (at worst $m = 0$), we have that for each $y \in p[B]$ $H(y)$ has fixed elements in $p[C]$.

We say y_1 is equivalent to y_2, if $hy_1 = y_2$ for some $h \in H$. Let S be a maximal set of pairwise non-equivalent elements of $p[B]$. Let g be a function from S into $p[C]$ such that gy is fixed under $H(y)$ for all $y \in S$. Let $x_1 = \{\langle hy, h(gy) \rangle : y \in S \text{ and } h \in H\}$. x_1 represents a function: If $hy = h'y$, then $h^{-1}h' \in H(y) \subseteq H(gy)$, whence $h(gy) = h'(gy)$; $\text{dom}(x_1) = p[B]$ because of the maximality of S and the invariance of $p[B]$ (see (12)). Also $\text{rg}(x_1) \subseteq p[C]$. Thus x_1 is an H-invariant function from $p[B]$ into $p[C]$.

Recall that the elements of $p[B \rightarrow C]$ are functions with domain $S(B)$, the set of 'possible proofs of B'. Let x_2 be any invariant function from $S(B) - p[B]$ into $S(C)$, for instance the constant function with an invariant element of $S(C)$ as value (which is easily seen to exist, since both Γ and Π

contain the σ-invariant element c_0). Then $x = x_1 \cup x_2$ belongs to $p[B \to C]$ and is invariant under σ^n.

4. Let A be $\forall v B$. '*only if*': Let $x \in p[\forall v B]$, $\sigma^n x = x$. Let $s_n R s$, $d \in \Psi(s)$. As before, we have $n|m$ and $s = s_m$ for $m = \varphi(s)$. By (8) there is $c \in \Gamma_m$ such that $c^- = d$. By (10) $\sigma^m c = c$. Therefore $\sigma^m(xc) = xc$. Also $xc \in p[B^v_c]$ and $B^v_c \in F(\Gamma_m)$. Induction hypothesis gives $\Phi((B^v_c)^-, s) = T$. $(B^v_c)^-$ is $(B^-)^v_d$. Hence $\Phi(\forall v B^-, s_n) = T$.

'*if*': Let $\Phi(\forall v B^-, s_n) = T$. Let $c \in \Gamma$. If H is the group generated by σ^n, then $H(c)$ is generated by σ^m for some m with $n|m$. By (10) $c \in \Gamma_m$. By (7) $c^- \in \Psi(s_m)$. By (4) $s_n R s_m$. Therefore $\Phi((B^v_c)^-, s_m) = T$. Also $B^v_c \in F(\Gamma_m)$. By induction hypothesis $H(c)$ has fixed elements in $p[B^v_c]$. As before we get an invariant function. There is no trouble with the range, since for all $h \in H$, $h(p[B^v_c]) = p[B^v_c] = p[B^v_{hc}]$ in virtue of (12) and (11). Therefore σ^n has fixed elements in $p[\forall v B]$.

5. Let A be $\exists v B$. The proof is straightforward, using (7), (8) and (10).

This concludes the proof of the lemma.

Proof of part (2) of the theorem:

Let A be a closed formula containing no individual constants other than c_0. Then $A \in F(\Gamma_1)$ and $A^- \in F(\Psi(s_1)) = F(\Psi(A))$. Assume not $\vdash A$. Lemma 1 gives a model Φ such that $\Phi(A^-, A) = F$. Let p be the proof assignment associated to Φ. Then by lemma 2, σ has no fixed element in $p[A]$. Therefore $p[A]$ contains no invariant functional.

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EXTENDING THE TOPOLOGICAL INTERPRETATION TO INTUITIONISTIC ANALYSIS, II

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This paper is a sequel to the paper [12] written for Professor Heyting under the same title. Nearly all of the questions left open in [12] have been answered. In particular the results of section 3 in [12] having to do with universal formulae of the theory of $<$ in *three* variables have been extended to *arbitrary* universal formulae in section 5. (Our numbering of sections continues that of [12].) We then discuss in section 6 the general metamathematical implications of the method of section 5 for the theory of the topological model of intuitionistic analysis. In section 7 the important step is taken of enlarging the model to encompass arbitrary (extensional) real functions. The main result is the verification in the model of Brouwer's theorem on continuity: *all functions are uniformly continuous on closed intervals*. The proof is given in detail along with several related results. (The reader will have to refer to [12] for notation and the definition of the model.)

The author was thus able to conclude this paper feeling that he had a rather good grasp of the basic properties of the real numbers of the model. Several further projects remain to be carried out, however. The next important step is to discuss the corresponding topological interpretation of *second-order arithmetic* and the theory of free-choice sequences of integers. This will make possible an exact comparison of the theory of the model and the usual axiomatic theories of intuitionistic analysis (which will no doubt be one of the main topics of part III of this series of papers.) Following such work it is obvious that attention must be given to obtaining a constructive version of the model. Kreisel has suggested that the theory of constructive and lawless sequences (the system of [8]) may provide the proper framework

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