# 1 Natural deduction and Kripke Semantics

## 1.1 Relation between IQC and CQC.

There are several logic systems:

- CQC Classical predicate logic
- IQC Intuitionistic predicate logic. This is CQC without $\bot_c$.
- MQC Minimal predicat logic. This is IQC without $\bot_i$.

We are going to look for a class F such that for all formulas A in this class $CQC \vdash A \Leftrightarrow IQC \vdash A$.

**Definition** A formula A is negative (not a negation) if every prime P that occurs in A only occur negated and A does not contain $\vee$ or $\exists$.

**Lemma 1.** *$MQC \vdash A \leftrightarrow \neg\neg A$ if A negative.*

**Definition** Let A be a formula of a predicate logic system. The (Gödel Gentzen) negative translation $^g$ is defined inductively by:

- $\bot^g := \bot$
- $P^g := \neg\neg P$
- $(A \wedge B)^g := A^g \wedge B^g$
- $(A \longrightarrow B)^g := A^g \longrightarrow B^g$
- $(\forall x A)^g := \forall x A^g$
- $(A \vee B)^g := \neg(\neg A^g \wedge \neg B^g)$
- $(\exists x A)^g := \neg\forall x \neg A^g$

**Theorem 2.** *For all A:*

- $\vdash_c A \leftrightarrow A^g$
- $\Gamma \vdash_c A \iff \Gamma^g \vdash_m A^g$

*With* $\Gamma^g = \{B^g : B \in \Gamma\}$

**Corollary 3.** *For all negative A, $CQC \vdash A$ iff $IQC \vdash A$.*

## 1.2 Kripke Semantics

We discuss Kripke Semantics for Pure Predicate Logic:

**Definition** A Kripke Model for IQC is a quadruple $M \equiv (K, \leq, D, \Vdash)$, such that

- $(K, \leq)$ is an inhabited poset.
- $D$ is a monotone function assigning inhabited sets to the elements of $K$.
- $\Vdash$ is a relation from K to the prime formulas of the extended language $L \cup (\bigcup\{D(k) \mid k \in K\})$, such that

$$k \Vdash R^n(d_1, ..., d_n) \implies d_i \in D(k), \text{ forall } i \leq n$$

$$k \Vdash R^n(d_1, ..., d_n) \wedge k \leq k' \implies k' \Vdash R^n(d_1, ..., d_n).$$

We extend the definition of $\Vdash$ to all sentences $A$ as follows:

- $k \Vdash A \wedge B \iff k \Vdash A$ and $k \Vdash B$,
- $k \Vdash A \vee B \iff k \Vdash A$ or $k \Vdash B$,
- $k \Vdash A \rightarrow B \iff \forall k' \geq k,$ if $k' \Vdash A$ then $k' \Vdash B$,
- not $k \Vdash \perp$.
- $k \Vdash \forall x A(x) \iff \forall k' \geq k \forall d \in D(k')(k' \Vdash A(d))$
- $k \Vdash \exists x A(x) \iff \exists d \in D(k)(k \Vdash A(d))$

**Theorem 4.** *(Soundness for pure IQC).* $\Gamma \vdash A \implies \Gamma \Vdash A$.

**Definition** Let C be a set of constants. A set of sentences $\Gamma$ in the language $L$ is called $C$-seturated iff:

- $\Gamma$ is consistent, i.e. there is no finite $\Gamma_0 \subset \Gamma$ such that $\vdash \neg(\bigwedge \Gamma_0)$.
- $\Gamma \vdash A \implies A \in \Gamma$,
- $\Gamma \vdash A \vee B \implies \Gamma \vdash A$ or $\Gamma \vdash B$,
- $\Gamma \vdash \exists x A(x) \implies$ for some $c \in C$, $A(c) \in \Gamma$.

**Lemma 5.** *(saturation lemma). Suppose $\Gamma \nvdash A$, $\Gamma$, $A$ in a language $L$. Let $C = \{c_0, c_1, ...\}$ be a countable set of constants not in $L$, and let $L(C)$ be $L$ extended with $C$. Then there is a $C$-saturated $\Gamma^\omega \supset \Gamma$, such that $\Gamma^\omega \nvdash A$.*

**Theorem 6.** *(strong completeness for IQC).* $\Gamma \Vdash A \implies \Gamma \vdash A$.