1 Introduction to Realizability

The material treated in this and the subsequent sections is based mainly on (Troelstra & Van Dalen 1988, ch. 3–4) and (Streicher 2007/08).

1.1 Motivation

We have already encountered several 'models of intuitionism'. But where is the intuition?

"The original purpose for having the models appears to have been for obtaining independence or consistency results for certain formalizations of intuitionism. . . [O]f course, if the models could be also justified as being plausible interpretations of intuitionistic thinking, so much the better." (López-Escobar 1981)

The *BHK interpretation* provides us with an informal interpretation of intuitionism.

- A proof of $P \wedge Q$ is a pair (a, b), where a is a proof for P and b is a proof for Q.
- A proof of $P \to Q$ is a function f that converts a proof for P into a proof for Q.
- A proof for $\exists x \in S : P(x)$ is a pair (a, b), where $a \in S$ and b is a proof for $P(a)$.

Realizability provides us with a formal apparatus with which to explicate these informal criteria.

1.2 Heyting Arithmetic

Definition 1.1. *Heyting Arithmetic*, or **HA**, is the **IQC**-theory, formulated in the language with equality, constant symbol 0 and symbols for all primitive recursive functions, which contains the defining equations for the primitive recursive recursive functions as well as the Peano axioms.

Proposition 1.2. *In* **HA***, we can define* $A \vee B$ *as* $\exists x((x = 0 \to A) \wedge (x \neq 0) \to B)$.

1.3 Kleene Realizability

Let $\langle . \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a paring function. Denote the first and second components of a number n as $\mathrm{fst}(n)$ and $\mathrm{snd}(n)$ respectively.

Definition 1.3. Let $n \in \mathbb{N}$. Then:

- n realizes $t = s$ iff $t = s$,
- n realizes $A \wedge B$ iff fst(n) realizes A and snd(n) realizes B,
- n realizes $A \vee B$ iff fst$(n) = 0 \Rightarrow$ snd(n) realizes A and fst$(n) \neq 0 \Rightarrow$ snd(n) realizes B,
- n realizes $A \to B$ iff for every m realizing A, $\varphi_n(m)$ is defined and $\varphi_n(m)$ realizes B,
- n realizes $\exists x A(x)$ iff snd(n) realizes $A(\text{fst}(n))$,
- n realizes $\forall x A(x)$ iff for all m: $\varphi_n(m)$ is defined and $\varphi_n(m)$ realizes $A(m)$.

Example 1.4. *Is* $\forall x(\exists y(2y = x) \vee \neg\exists y(2y = x))$ *realizable?*

In practice, it is advantageous to represent the realizability relation syntactically:

Definition 1.5. We define the predicate rn as follows:

- n rn $P \equiv P$, for P atomic,
- n rn $A \wedge B \equiv$ fst(n) rn $A \wedge$ snd(n) rn B,
- n rn $A \vee B \equiv$ (fst(n)$= 0 \to$ snd(n) rn A) $\wedge$ (fst(n)$\neq 0 \to$ snd(n) rn B),
- n rn $A \to B \equiv \forall m(m \text{ rn } A \to \varphi_n(m) \text{ rn } B \wedge \varphi_n(m) \downarrow)$,
- n rn $\exists x A(x) \equiv$ snd(n) rn $A(\text{fst}(n))$,
- n rn $\forall x A(x) \equiv \forall m(\varphi_n(m) \text{ rn } A(m) \wedge \varphi_n(m) \downarrow)$

where n is not free in A.

1.4 Soundness

Theorem 1.6. *For every sentence A, if* ***HA*** $\vdash A$*, then there exists a term e such that* ***HA*** $\vdash e$ *rn* A*.*

1.5 Church's Thesis

Theorem 1.6 is nice, but we want more. However, danger looms. Consider *Church's thesis*:

$$\forall n \exists m A(n, m) \to \exists e \forall n(A(n, \varphi_e(n))) \tag{CT$_0$}$$

This schema can be seen to spell trouble for a completeness theorem. This warrants an expansion of **HA**.

1.6 Characterisation of Number Realizability

We will often adept the following notation: If $\lambda x.F(x)$ is some partial recursive function with code c, then $[\lambda x.F(x)] := c$. Hence for A some formula we have $[\lambda x.F(x)]$ **rn** $A = c$ **rn** A and $\varphi_{[\lambda x.F(x)]} = \varphi_c$. If no confusion will arise we sometimes ommit the $[]$.

Definition 1.7. A formula A is called *almost negative* or *essentially* $\exists$*-free* if it is build from atomic fomulas and existentially quantified atomic formulas (i.e. $\exists x, t = s$) using only $\wedge$, $\rightarrow$ and $\forall$.

Proposition 1.8. *For almost negative formulas A it holds that*

1. $\mathbf{HA} \vdash \exists x(x \,\textbf{\textit{rn}}\, A) \rightarrow A$.
2. *There is a term ψ_A with* $\mathbf{HA} \vdash A \rightarrow \psi_A \,\textbf{\textit{rn}}\, A \wedge \psi_A \downarrow$.

Therefore $\mathbf{HA} \vdash A \leftrightarrow \exists x(x \,\textbf{\textit{rn}}\, A)$.

Proposition 1.9. *For all A we have that x **rn** A is equivalent to an almost negative formula.*

Proposition 1.10. *For every formula A it hold that*

$$\mathbf{HA} \vdash \exists y(y \,\textbf{\textit{rn}}\, A) \leftrightarrow \exists x(x \,\textbf{\textit{rn}}\, \exists y(y \,\textbf{\textit{rn}}\, A)).$$

Now we can formulate the Extended Church Thesis.

Definition 1.11. The *Extended Chuch Thesis* is the following schema, where A is almost negative:

$$\mathrm{ECT}_0 \qquad \forall x(A(x) \rightarrow \exists y B(x,y)) \rightarrow \exists e \forall x(A(x) \rightarrow B(x, \varphi_e(x)) \wedge \varphi_e(x) \downarrow).$$

Proposition 1.12. *For every instance F of* ECT_0 *we have* $\mathbf{HA} \vdash \exists x(x \,\textbf{\textit{rn}}\, F)$.

Theorem 1.13. *For all formulas A it holds that*

- $\mathbf{HA} + \mathrm{ECT}_0 \vdash A \leftrightarrow \exists x(x \,\textbf{\textit{rn}}\, A)$.
- $\mathbf{HA} + \mathrm{ECT}_0 \vdash A \iff \mathbf{HA} \vdash \exists x(x \,\textbf{\textit{rn}}\, A)$.

References

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[2] Streicher, Thomas (2007/08). *Realizability*. Lecture notes.

[3] Troelstra, A.S. & D. van Dalen (1988). *Constructivism in Mathematics: An Introduction*. Amsterdam: North-Holland.