Seminar Models of Intuisionism

Handout lecture 7: Calculus of problems

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Definition: A *problem* is some exercise for us to identify certain desirable elements in a set. Given problems A and B, we also have problems $A \wedge B$: this is the problem of solving both A and B, $A \vee B$: this is the problem of solving at least A or B, $A \to B$: this is the problem of solving B given a solution to A, and $\neg A$, this is the problem of deriving a contradiction from a solution to A. Problems of this form will be called *composite problems*. We say that a composite problem is *solvable* if we can find a solution to it which is independent of the problems A and B.

Assumption: We assume that the following problems have been solved, for all A, B, C:

- $A \to A \wedge A$
- $A \wedge B \to B \wedge A$
- $(A \to B) \to (A \wedge C \to B \wedge C)$
- $B \to (A \to B)$
- $A \wedge (A \to B) \to B$
- $A \to A \vee B$
- $A \vee B \to B \vee A$
- $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$
- $\neg A \to (A \to B)$
- $(A \to B) \wedge (A \to \neg B) \to \neg A$

We also assume the inference rules of modus ponens, substitution and solving A from $A \wedge B$.

Claim (Kolmogorov): This notion of solvability coincides with provability in intuitionistic propositional logic.

Formalized notion: Fix a countably infinite set of 'elementary problems'. If A is a problem, denote the set of possible solutions of A by $F(A)$, and the set of actual solutions of A by $X(A)$. Then define for problems A and B:

- $F(A \wedge B) = F(A) \times F(B)$ and $X(A \wedge B) = X(A) \times X(B)$.
- $F(A \vee B) = F(A) \sqcup F(B)$ and $X(A \vee B) = X(A) \sqcup X(B)$.

- $F(A \to B) = F(B)^{F(A)}$ and $X(A \to B) = \{f \in F(B)^{F(A)} \mid f(X(A)) \subseteq X(B)\}$

We define $\neg A$ as $A \to A_0$, where A_0 is an elementary problem such that $X(A_0) = \emptyset$.

Definition: Suppose $A(a_1, ..., a_n)$ is a problem. Then we call A solvable for the system $F(a_1), ..., F(a_n)$ if there is an element in $F(A)$ which is in every $X(A)$, so for every possible assignment $X(a_1), ..., X(a_n)$. If A is solvable for all $F(a_1), ..., F(a_n)$, then we call it *identically solvable.*

Minimal logic: Minimal logic has the following axioms:

- $x \to (y \to x)$
- $(x \to (y \to z)) \to ((x \to y) \to (x \to z))$
- $x \to (y \to (x \wedge y))$
- $x \wedge y \to x$
- $x \wedge y \to y$
- $x \to (x \vee y)$
- $y \to (x \vee y)$
- $(x \to z) \to ((y \to z) \to ((x \vee y) \to z))$

And we also have modus ponens and substitution.

Proposition: All these axioms are identically solvable.

Lemma 1: If $\Delta \vdash A$ and every formula in Δ is identically solvable, then A is identically solvable.

Definition: A *critical implication* is a formula of the form

$$\bigwedge_{i<n} ((P_i \to Q_i) \to Q_i) \to R$$

Where each P_i is a nonempty elementary conjunction, R and each Q_i are nonempty elementary disjunctions, and for each i, there is no variable occurring in both P_i and Q_i.

Lemma 2: Every critical implication is refutable.

Lemma 3: For every formula A, either $\vdash A$ or $A \vdash J$ with J a critical implication.

A proof of this lemma can be found in [4].

Theorem: If A is identically solvable, then it is derivable.

References

[1] A. Kolmogorov. *On the interpretation of intuitionistic logic.* Math. Zeitschrift 35, 1932, pages 58-65. Translated to English by James McKinna, Edingburgh 2014.

[2] Ju. T. Medvedev. *Finite problems.* Soviet Math Doklady, 1962, no. 1. Translated to English by Elliott Mendelson.

[3] Ju. T. Medvedv. *Interpretation of logical formulas by means of finite problems and its relation to the realizability theory.* Soviet Math Doklady, 1963, no. 1. Translated to English by Sue Ann Walker.

[4] A. V. Chernov et al. *Variants of realizability for propositional formulas and the logic of the weak law of excluded middle.* Bradfield J. (eds) Computer Science Logic. CSL 2002. Lecture Notes in Computer Science, vol 2471. Springer, Berlin, Heidelberg.