Seminar Constructible Sets

Handout session 3: Chapter I, section 9

2018-03-07

Today we will construct a language $\mathcal{L}_V$, an internal analog of LST, whose symbols and formulas are given by sets. We will show that properties of this language have low Lévy complexity, even with respect to certain systems weaker than ZFC.

Remark. We use Δ_0 to denote what Devlin calls Σ_0 (or Π_0, they are all the same).

1 Encoding $\mathcal{L}_V$

Definition 1. A *sequence* is a function whose domain is an ordinal α. A sequence is finite iff $\alpha \in \omega$.

We denote a sequence of values $x_0, \dots, x_{n-1}$ (with domain n) by $\langle x_0, \dots, x_{n-1} \rangle$. We denote the concatenation of two sequences s and t by $s \frown t$. Given a sequence s with domain n, we denote the greatest element $n-1$ of n by $\|s\|$.

We can can express that a sequence is finite using a Δ_0 formula Finseq.

Our abbreviation scheme can be summarised as follows:

the n^{th} variable, x_n	$(2, n)$
the set x	$(3, x)$
$x \in y$	$\langle 0, 4, x, y, 1 \rangle$
$x = y$	$\langle 0, 5, x, y, 1 \rangle$
$\phi \wedge \psi$	$\langle 0, 6 \rangle \frown \phi \frown \psi \frown \langle 1 \rangle$
$\neg\phi$	$\langle 0, 7 \rangle \frown \phi \frown \langle 1 \rangle$
$\exists u \phi$	$\langle 0, 8, u \rangle \frown \phi \frown \langle 1 \rangle$

Properties such as "θ is the negation of ϕ" can be encoded as formulas of LST. The encoding of the above ($F_\neg(\theta, \phi)$) is

$$\begin{aligned}&\mathrm{Finseq}(\theta) \wedge \mathrm{Finseq}(\phi) \wedge [\mathrm{dom}(\theta) = \mathrm{dom}(\phi) + 3] \wedge [\phi(0) = 0] \\ &\wedge [\theta(1) = 7] \wedge [\theta(\|\theta\|) = 1] \wedge (\forall i \in \mathrm{dom}(\phi))[\theta(i+2) = \phi(i)].\end{aligned}$$

Similarly, there exist formulas $\mathrm{Vbl}(x)$, $\mathrm{Const}(x)$, $F_=(\theta, x, y)$, $F_\in(\theta, x, y)$, $F_\wedge(\theta, \phi, \psi)$, $F_\exists(\theta, u, \phi)$, $\mathrm{PFml}(\phi)$. We can relativize these properties with respect to a set u, giving the language L_u; we require that constants come from u.

The formulas $\mathrm{Fml}(\phi)$ ("ϕ is a formula of $\mathcal{L}_V$") and $\mathrm{Sat}(u, \phi)$ ("ϕ is a sentence of $\mathcal{L}_u$ and is true in the structure u") are of particular interest to us.

In the remainder of the talk, we will look at the Lévy complexity of these formulas. Devlin claims that Fml and Sat are Σ_1, while $\mathrm{Vbl}(x)$, $\mathrm{Const}(x)$, etc. are Δ_0. The latter is not true: $F_\wedge$ is not Δ_0, but it is Δ_1, making the absoluteness results still go through.

2 Various systems

In the following definition we keep the naming from [2], and present the axioms in the same style. So V denotes the universe for the system (it should be clear that everything can be expressed in first-order logic).

Definition 2. We define three different systems, where **BS** will be equivalent to Devlin's *Basic Set Theory* (see [1, page 36]).

ReS The simplest system we discuss is **ReS**, which is given by the following axioms.

Extensionality — If two sets contain the same elements, they are equal:

$$[\forall x \in a(x \in b) \wedge \forall x \in b(x \in a)] \rightarrow a = b$$

Empty set — There is an empty set:

$$\emptyset \in V$$

Pairing — Any two sets can be paired to form a new set:

$$x, y \in V \rightarrow \{x, y\} \in V$$

Union — One can form the union of all the sets in a set:

$$x \in V \rightarrow \bigcup x \in V$$

Δ_0-separation — For every Δ_0-class A we have

$$x \in V \rightarrow x \cap A \in V,$$

that is, if $\phi(y)$ is Δ_0 and x is a set, then $\{y \in x : \phi(y)\}$ is a set.

Π_1-foundation — Every non-empty Π_1-class A contains a set x that is disjoint from A. That is, if A is a non-empty class that is given by a Π_1-formula, then

$$\exists x \in A(x \cap A = \emptyset).$$

BS The system **BS** from Devlin is actually equivalent to **ReS** with the following three axioms added

Cartesian product — The Cartesian product[1] of any two sets exists:

$$x, y \in V \rightarrow x \times y \in V$$

Full foundation — Every non-empty class A contains a set x that is disjoint from A. That is, if A is a non-empty class, then:

$$\exists x \in A(x \cap A = \emptyset)$$

Note that this axiom is a strengthening of Π_1-foundation, since we allow A now to be any non-empty class.

[1]It should be noted that **ReS** already allows for creating tuples in the usual way: $(y_1, y_2) = \{y_1, \{y_1, y_2\}\}$ and $(y_1, y_2, y_3) = (y_1, (y_2, y_3))$ and so on.

Infinity There is an infinite set:

$$\omega \in V$$

MW The final system we will discuss is **MW**, which is **ReS** with the Cartesian product axiom and the following axiom:

$$\forall a \forall n \in \omega([a]^n \in V),$$

where $[a]^n$ denotes the class of all subsets of a of size n.

3 Repairing Devlin's results

After each lemma we have included the number is has in [1] or [2]. The lemmas with a 9 are from [1], and those with a 10 are from [2]. Note that we have also included the lemmas from [1] that are false (but we have explicitly stated it when they are false).

The lemmas will be about the following formulas.

Definition 3. We only give the meaning of the formulas, for their exact definition we refer to [1] and [2].

$\mathrm{Seq}(u, a, n)$ "u is the set of all sequences in a of length $< n$"

$\mathrm{Fml}(x)$ "x is an $\mathcal{L}$-formula"

$\mathrm{Sat}(u, \phi)$ "ϕ is an $\mathcal{L}_u$-sentence that is true in the structure $\langle u, \in \rangle$"

Lemma 1 (Lemma 9.5)**.** $\mathrm{Seq}(u, a, n)$ *is* $\Delta_1^{\mathbf{BS}}$ ***(false)****.*

Lemma 2 (Lemma $10 \cdot 10$)**. ReS** *proves that if a is finite, then for all natural n there is u such that* $\mathrm{Seq}(u, a, n)$*. It follows that* **ReS** *proves that for all finite a, for all natural n one has*

$$\mathrm{Seq}(u, a, n) \longleftrightarrow \forall v(u \neq v \to \neg \mathrm{Seq}(v, a, n)).$$

Lemma 3 (Lemma 9.6)**.** $\mathrm{Fml}(x)$ *is* $\Delta_1^{\mathbf{BS}}$*. In [2] this is actually sharpened to:* $\mathrm{Fml}(x)$ *is* $\Delta_1^{\mathbf{ReS}}$*.*

Following a very similar process one can define a formula $\mathrm{Fml}(x, u)$ which means "x is an $\mathcal{L}_u$-formula", and we obtain the following lemma.

Lemma 4 (Lemma 9.7)**.** $\mathrm{Fml}(x, u)$ *is* $\Delta_1^{\mathbf{ReS}}$*.*

Lemma 5 (Lemma 9.10)**.** $\mathrm{Sat}(u, \phi)$ *is* $\Delta_1^{\mathbf{BS}}$ ***(false)****.*

The following lemma has no actual number, but is shown in [2, page 44] in the subsection called "The cure in **MW**".

Lemma 6. $\mathrm{Sat}(u, \phi)$ *is* $\Delta_1^{\mathbf{MW}}$*.*

4 Exercises

Exercise 1. In **ZF** we 'only' have the axiom of "set foundation", that is:

$$\forall x(x \neq \emptyset \to \exists y \in x(x \cap y = \emptyset)).$$

In **BS** we have the axiom of "full foundation", which may seem stronger. Prove that "full foundation" is a derivable in **ZF** (hint: you will want to use the transitive closure of a set described in [1, page 12]).

Exercise 2. Let T be a theory, $n \in \mathbb{N}$ and let $\phi(x)$ be a Σ_n formula such that

$$T \vdash \exists x(\phi(x))$$
$$T \vdash \phi(x) \leftrightarrow \forall y(\phi(y) \to x = y)$$

Show that $\phi(x)$ is Δ_n^T.

Exercise 3. An attempt at integer addition for $n, m \in \mathbb{N}$ is a function $A : \omega \times \omega \to \omega$ such that for all $n' \leq n$ and $m' \leq m$, $A(n', m') = n' + m'$.

Show that the property "A is an attempt at integer addition for n, m" can be expressed as a Δ_0 formula. (You may use lemma 8.4 from [1].)

References

[1] Keith J. Devlin, *Constructibility*, Springer-Verlag Berlin, ISBN 0-387-13258-9, 1984.

[2] A. R. D. Mathias, *Weak systems of Gandy, Jensen and Devlin*, Centre de Recerca Matemàtica, Bellaterra, Catalonia and ERMIT, Université de la Réunion, 2006.