Handout Chapter I section 11
Chapter II section 1-2

Seminar Constructible Sets

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Reminder (lemma I.9.11)**.** Let $\Phi(\vec{x})$ be a Σ_0 formula of LST and let $\varphi(\vec{x})$ be its counterpart in $\mathscr{L}$, then

$$\mathsf{ZF} \vdash \forall u(\forall \vec{x} \in u)\left[\Phi^u(\vec{x}) \leftrightarrow \vDash_u \varphi\left(\mathring{\vec{x}}\right)\right]$$

Reminder (lemma I.9.15)**.** Let $\Phi(\vec{x})$ be a Σ_0 formula of LST and let $\varphi(\vec{x})$ be its counterpart in $\mathscr{L}$, then

$$\mathsf{ZF} \vdash \text{“For } M \text{ a transitive set, } (\forall \vec{x} \in M)\left[\Phi(\vec{x}) \leftrightarrow \vDash_M \varphi\left(\mathring{\vec{x}}\right)\right]\text{”}$$

1.11) Kripke-Platek Set Theory & Admissible sets

Definition 1. KP (Kripke-Platek set theory) is the theory given by the axioms of BS enriched with Δ_0-collection:

(i) Axiom of Extensionality
(ii) Axiom of Induction
(iii) Axiom of Pairing
(iv) Axiom of Union
(v) Axiom of Infinity
(vi) Axiom of Cartesian Product
(vii) Axiom schema of Δ_0-Comprehension
(viii) Axiom schema of Δ_0-Collection

Reminder (amenable sets)**.** A set M is **amenable**, if it is transitive and satisfies

(i) $(\forall x \in M)(\forall y \in M)(\{x, y\} \in M)$
(ii) $(\forall x \in M)\,(\bigcup x \in M)$
(iii) $\omega \in M$
(iv) $(\forall x \in M)(\forall y \in M)(x \times y \in M)$
(v) If $R \subseteq M$ is $\Sigma_0(M)$, then $(\forall x \in M)(R \cap x \in M)$

Definition 2. A set M is **admissible**, if M is amenable and satisfies

(vi) If a relation $R \subseteq M \times M$ is $\Sigma_0(M)$ and $(\forall x \in M)(\exists y \in M)(y\ R\ x)$, then for any $u \in M$ there is a $v \in M$ such that $(\forall x \in u)(\exists y \in v)(y\ R\ x)$

Lemma 3. Let $\Phi(x, y, \vec{p})$ be Σ_1, and $\Psi(z, \vec{p})$ be Δ_1^{KP} then

$$\mathsf{KP} \vdash \forall \vec{p}\,[\forall x \exists y \Phi(x, y, \vec{p}) \to \forall u \exists v (\forall x \in u)(\exists y \in v)\Phi(x, y, \vec{p})] \qquad (\Sigma_1\text{-Collection})$$
$$\mathsf{KP} \vdash \forall \vec{p}\,[\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \Psi(z, \vec{p}))] \qquad (\Delta_1\text{-Comprehension})$$

Lemma 4 (Recursion Theorem)**.** Let G be a total $(n+2)$-ary Σ_1^{KP} function over V. Then there is a total $(n+1)$-ary Σ_1^{KP} function F over V such that

$$\mathsf{KP} \vdash F(\vec{x}, y) = G(\vec{x}, y, (F(\vec{x}, z) \mid z \in y)).$$

2.1) Definition of the Constructible Universe

Reminder (definition of x-definable)**.** A subset y of x is x**-definable** iff there is a formula $\varphi(v)$ of $\mathscr{L}_x$ such that $y = \{a \in x \mid \vDash_x \varphi(\mathring{a})\}$. Define $\mathrm{Def}(x)$ to be the set of x-definable subsets of x.

$\mathrm{Def}(u)$ can be defined in LST as:

$$\begin{aligned} v = \mathrm{Def}(u) \quad \leftrightarrow \quad & (\forall x \in v)(\exists \varphi)\Big[\mathrm{Fml}(\varphi, u) \wedge \mathrm{Fr}(\varphi, \{v_0\}) \\ & \wedge\, x = \big\{z \in u \mid (\exists \psi)(\mathrm{Sub}(\psi, \varphi, v_0, \mathring{z}) \wedge \mathrm{Sat}(u, \psi))\big\}\Big] \\ & \wedge\, (\forall \varphi)\Big[(\mathrm{Fml}(\varphi, u) \wedge \mathrm{Fr}(\varphi, \{v_0\})) \\ & \rightarrow (\exists x \in v)(x = \big\{z \in u \mid (\exists \psi)(\mathrm{Sub}(\psi, \varphi, v_0, \mathring{z}) \wedge \mathrm{Sat}(u, \psi))\Big] \end{aligned}$$

Definition 5. By recursion on $\alpha \in \mathrm{On}$ we define

$$L_0 = \varnothing, \qquad L_{\alpha+1} = \mathrm{Def}(L_\alpha), \qquad L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha \text{ if } \gamma \text{ is limit}, \qquad L = \bigcup_{\alpha \in On} L_\alpha.$$

The set L is the **constructible universe**. A set x is **constructible** iff $x \in L$.

Lemma 6. The constructible hierarchy has the following properties:
(i) $\alpha \leq \beta$ implies $L_\alpha \subseteq L_\beta$.
(ii) L_α is transative for each α. (L is transitive).
(iii) $L_\alpha \subseteq V_\alpha$ for all α.
(iv) $\alpha < \beta$ implies $\alpha \in L_\beta$ and $L_\alpha \in L_\beta$. ($\mathrm{On} \subseteq L$).
(v) $L \cap \alpha = L_\alpha \cap \mathrm{On} = \alpha$ for all α.
(vi) $L_\alpha = V_\alpha$ for $\alpha \leq \omega$.
(vii) $|L_\alpha| = |\alpha|$ for $\alpha \geq \omega$.

Definition 7. Let M be a transitive proper class and T a theory in LST, then M is an **inner model** of T if for every axiom Φ of T we have $T \vdash \Phi^M$.

Theorem 8. The class L is an inner model of ZF. In fact $\mathsf{ZF} \vdash (\mathsf{AoC})^L$ (this will be treated next week).

2.2) Constructibility

Lemma 9. L_α is amenable for each uncountable limit ordinal α.

Reminder (uniformly $\Delta_n^{\mathbf{M}}$)**.** Given a family of classes $\mathscr{F}$ and a class $\mathcal{A} = \{\vec{x} \mid \Phi(\vec{x})\}$ of m-tuples defined by LST formula Φ, we say $\mathcal{A}$ is uniformly $\Delta_n^{\mathbf{M}}$ if there are is a Σ_n formula $\varphi_1(\vec{x})$ and Π_n formula $\varphi_2(\vec{x})$ of $\mathscr{L}$ such that for each $\mathbf{M} \in \mathscr{F}$ we have $A \cap M^m = \Big\{\vec{x} \mid \vDash_{\mathbf{M}} \varphi_i\Big(\mathring{\vec{x}}\Big)\Big\}$.

Remark 10. Devlin states that the class defined by Sat (that is $\{(u, \varphi\} \mid \mathrm{Sat}(u, \varphi)\}$) is uniformly Δ_1^M for amenable sets M. This is **false**, as Mathias gives a countermodel. It can be fixed by defining $\mathcal{S}(x)$ to be the set of finite subsets of x, and letting M be an $\mathcal{S}$-amenable set, which is an amenable set such that $x \in M$ implies $\mathcal{S}(x) \in M$. Now the class Sat is $\Delta_1^{M^{\mathcal{S}}}$ for $\mathcal{S}$-amenable sets $M^{\mathcal{S}}$.

Definition 11. For an LST formula $\Phi(\vec{x})$, let $(*)$ and $(**)$ be the following properties:
$(*)$ $\Phi(\vec{x})$ is Δ_1^{KP}.

(∗∗) The class $\{\vec{x} \mid \Phi(\vec{x})\}$ is uniformly $\Delta_1^{L_\alpha}$, for all uncountable limit ordinals α.

We define the following LST formulas:

$$\begin{aligned}
\mathrm{Seq}(y,x): \quad & \exists f\Big[\text{“}f \text{ is a function”} \wedge \mathrm{dom}(f)=\omega \wedge f(0)=\varnothing \wedge y = \bigcup \mathrm{ran}(f) \\
& \wedge (\forall n\in\omega)(\forall s\in f(n+1))(\exists t\in f(n))(\exists a\in x)\Big(s = t\cup\{(a,n)\}\Big) \\
& \wedge (\forall n\in\omega)(\forall s\in f(n))(\forall a\in x)(\exists t\in f(n+1))\Big(t = s\cup\{(a,n)\}\Big)\Big] \\
\mathrm{Pow}(y,x): \quad & \exists z\Big[\mathrm{Seq}(z,x)\wedge y = \{\mathrm{ran}(u)\mid u\in z\}\Big]
\end{aligned}$$

Lemma 12. $\mathrm{Seq}(y,x)$ satisfies (∗) and (∗∗).

Lemma 13. $\mathrm{Pow}(y,x)$ satisfies (∗) and (∗∗).

Lemma 14. There is an LST formula $D(v,u)$ such that $D(v,u) \leftrightarrow v = \mathrm{Def}(u)$. Moreover, $D(v,u)$ satisfies (∗) and (∗∗).

Lemma 15. There is an LST function $G(f,\alpha)$ such that G is true iff $f : \alpha+1 \to L$ sends $\beta \mapsto L_\beta$. Moreover, $G(f,\alpha)$ satisfies (∗) and (∗∗).

Let $H(x,\alpha)$ be the LST formula that says “$x = L_\alpha$”, which is defined as $H(x,a) \leftrightarrow \exists f[G(f,\alpha)\wedge (x = f(\alpha))]$.

Lemma 16. $H(x,\alpha)$ satisfies (∗) and (∗∗).

Lemma 17. If M is an inner model of KP, and α an ordinal, then $L_\alpha \in M$ and $(L_\alpha)^M = L_\alpha$. Hence $(L)^M = L$

Corollary 18.
- (i) If M is an admissible set and $\lambda = \sup(M\cap \mathrm{On})$, then for $\alpha<\lambda$ we have $(L_\alpha)^M = L_\alpha$. Hence $(L)^M = L_\lambda$.
- (ii) For any α we have $(L_\alpha)^L = L_\alpha$. Hence $(L)^L = L$.
- (iii) For $\alpha > \omega$ a limit ordinal and $\gamma<\alpha$ we have $(L_\gamma)^{L_\alpha} = L_\gamma$. Hence $(L)^{L_\alpha} = L_\alpha$.

Lemma 19. The LST formula “$x\in L$” (that is, x is constructible) is Σ_1^{KP}.

Definition 20. The **Axiom of Constructibility** (also *Hypothesis of Constructibility*) is the statement $\forall x(x\in L)$. Equivalently this means $V = L$.

Theorem 21. $\mathsf{ZF} \vdash (V=L)^L$.

1 Exercises

Exercise 1. Prove that the Axiom of Cartesian Product holds in the inner model L

$$\mathsf{ZF} \vdash (\forall x \forall y \exists z \forall u[u \in z \leftrightarrow (\exists a \in x)(\exists b \in y)(u = \langle a, b\rangle)])^L$$

without assuming II.1.2. However, you may assume that

$$\mathsf{ZF} \vdash \forall x \forall y \exists z \forall u[u \in z \leftrightarrow (\exists a \in x)(\exists b \in y)(u = \langle a, b\rangle)].$$

Exercise 2. Let $\mathrm{Add}(s, a, b)$ be the LST formula that states "$s = a + b$" $\wedge$ "a, b, s are natural numbers".

a) Prove that $\mathrm{Add}(s, a, b)$ is Δ_1^{KP}. $(*)$
b) Prove that the class defined by Add is uniformly $\Delta_1^{L_\alpha}$ for $\alpha > \omega$ limit. $(**)$