Solutions to Exercise sheet 5

Olle Torstensson & Tristan van der Vlugt

1 **Prove that the Axiom of Cartesian Product holds in the inner model L**

$$\mathsf{ZF} \vdash (\forall x \forall y \exists z \forall u[u \in z \leftrightarrow (\exists a \in x)(\exists b \in y)(u = \langle a, b\rangle)])^L$$

without assuming II.1.2. However, you may assume that

$$\mathsf{ZF} \vdash \forall x \forall y \exists z \forall u[u \in z \leftrightarrow (\exists a \in x)(\exists b \in y)(u = \langle a, b\rangle)].$$

Proof. Assume ZF.
Let $x, y \in L$. We want to show that there is a $z \in L$ such that for all $u \in L$,

$$u \in z \leftrightarrow (\exists a \in x)(\exists b \in y)(u = \langle a, b\rangle)$$

Let α be such that $x, y \in L_\alpha$. By assumption, cartesian products exist, so put $z = x \times y = \{u \mid (\exists a \in x)(\exists b \in y)(u = \langle a, b\rangle)\}$. By transitivity of L_α, we have that $x \subseteq L_\alpha$ and $y \subseteq L_\alpha$. Furthermore, for any $a, b \in L_\alpha$,

$$\begin{aligned}
\{a\} &= \{u \in L_\alpha \mid \models_{L_\alpha} u = \mathring{a}\} \in L_{\alpha+1};\\
\{a, b\} &= \{u \in L_\alpha \mid \models_{L_\alpha} u = \mathring{a} \vee u = \mathring{b}\} \in L_{\alpha+1};\\
\langle a, b\rangle &= \{\{a\}, \{a, b\}\} = \{u \in L_{\alpha+1} \mid \models_{L_{\alpha+1}} u = \mathring{\{a\}} \vee u = \mathring{\{a, b\}}\} \in L_{\alpha+2}.
\end{aligned}$$

Thus $x \times y \subseteq L_{\alpha+2}$ and then we have that

$$x \times y = \{u \in L_{\alpha+2} \mid \models_{L_{\alpha+2}} (\exists a \in \mathring{x})(\exists b \in \mathring{y})(u = \langle a, b\rangle)\} \in L_{\alpha+3} \subseteq L.$$

So $x \times y \in L$ and we have proven that:

$$\mathsf{ZF} \vdash [\forall x \forall y \exists z \forall u(u \in z \leftrightarrow (\exists a \in x)(\exists b \in y)(u = \langle a, b\rangle))]^L.$$

□

2 **Let** $\mathrm{Add}(s, a, b)$ **be the LST formula that states "**$s = a + b$**"** $\wedge$ **"**a, b, s **are natural numbers".**

a) **Prove that** $\mathrm{Add}(s, a, b)$ **is** Δ_1^{KP}**. (∗)**

Proof. The first step is to find a Σ_1 formula that describes $\mathrm{Add}(s, a, b)$. Surprisingly every submission of this homework had a different approach to this. Personally I would use the formula $\exists \Phi(s, a, b)$ with

$$\begin{aligned}
\Phi(s, a, b) :=&\text{"}f\text{ is a function"} \wedge \text{"}a,\ b,\ s\text{ are natural numbers"} \wedge \mathrm{dom}(f) = \langle a + 1, b + 1\rangle \wedge\\
&f(x, 0) = x \wedge f(x, y + 1) = f(x, y) + 1 \wedge f(a, b) = s
\end{aligned}$$

All the terms inside the brackets are Σ_0. A more compact function would be the unary f such that $\mathrm{dom}(f) = b + 1$, $f(0) = a$, $f(x + 1) = f(x) + 1$ and $f(b) = s$.
Clearly in KP we have

$$\mathsf{KP} \vdash \mathrm{Add}(s, a, b) \leftrightarrow \Phi(s, a, b)$$

and furthermore that f is unique for every triple s, a, b of natural numbers. By a previous homework exercise we can conclude that then there is a Π_1^{KP} formulation of Add □

b) **Prove that the class defined by Add is uniformly $\Delta_1^{L_\alpha}$ for $\alpha > \omega$ limit. (**)**

Proof. We wil prove that Add is uniformly $\Sigma_1^{L_\alpha}$ for $\alpha > \omega$ limit. That Add is $\Pi_1^{L_\alpha}$ as well follows from the uniqueness of s for each a, b and the fact that s is in L_α (it is a natural number).
Let add be the $\mathscr{L}$ equivalent of Add. We wish to show that

$$\mathrm{Add}(s, a, b) \;\Leftrightarrow\; \vDash_{L_\alpha} \mathrm{add}(\mathring{s}, \mathring{a}, \mathring{b})$$

As we saw in the lecture, if we want to show that Add is $\Sigma_1^{L_\alpha}$, the only case that is specific to Add is that $\exists f \Phi(f, s, a, b) \Rightarrow\ \vDash_{L_\alpha} \exists f \varphi(f, \mathring{s}, \mathring{a}, \mathring{b})$, where φ is the $\mathscr{L}$ equivalent of Φ. Therefore assume that $\exists f \Phi(f, s, a, b)$ for natural numbers s, a, b.
We know that $\omega \in L_\omega$, and therefore that $a, b, s \in L_\omega$. So for any two natural numbers n, m, we see that $\{n\}, \{n, m\} \in L_{\omega+1}$ and thus $\langle n, m\rangle \in L_{\omega+2}$. Each element of f is of the form $\langle n, \langle n, m\rangle\rangle$, and thus all elements in f are elements in $L_{\omega+4}$. This shows $f \in L_{\omega+5} \subseteq L_\alpha$. So if f is such that $\Phi(f, s, a, b)$, by uniqueness of f we see that $\vDash_{L_\alpha} \varphi(\mathring{f}, \mathring{s}, \mathring{a}, \mathring{b})$, which proves our claim. □