Seminar Constructible Sets

Model solution session 6: Chapter II, sections 3-5

2018-04-04

Exercise 1

In the proof of Lemma 8 (on the handout), ([1, Lemma 3.3(i)]), we give γ the value $\max(\omega, \alpha+5)$. Show why this value works for this proof.

Solution to exercise 1 (4 pt)

We know that the formula $WO(x, y)$ can be written as follows:

$$\exists\alpha[(x \in L_\alpha \wedge y \notin L_\alpha) \wedge \exists w(w = L_{\max(\omega,\alpha+4)} \wedge R(x, y, a, w))]$$

We now want to bind the quantifiers in this formula by L_γ for some γ. When we have that $\omega \leq \alpha$, so also that $\omega < \alpha + 5$, we can see that $\gamma = \alpha + 5$ suffices. We trivially have that $\alpha \in L_{\alpha+5}$ and since $w = L_{\alpha+4}$ we have that $w \in L_{\alpha+5}$.

In the case that $\alpha < \omega$, and therefore $\alpha + 5 < \omega$, we have to work a bit harder. We know that $WO(x, y) \leftrightarrow \neg((x = y) \wedge WO(y, x))$, and this last formula written out is as follows:

$$(x \neq y) \wedge \forall\alpha[(x \notin L_\alpha \vee y \in L_\alpha) \wedge \forall w(w \neq L_{\max(\omega,\alpha+4)} \vee \neg R(x, y, \alpha, w))]$$

When this formula holds, we also trivially have that the formula holds when you bind all the quantifiers by L_ω. So now assume that the formula holds, when you bind all the quantifiers by L_ω, that is, the following formula holds:

$$(x \neq y) \wedge \forall\alpha \in L_\omega[(x \notin L_\alpha \vee y \in L_\alpha) \wedge \forall w \in L_\omega(w \neq L_{\max(\omega,\alpha+4)} \vee \neg R(x, y, \alpha, w))]$$

We then know have that $\forall\alpha \in L_\omega(x \notin L_\alpha \vee y \in L_\alpha)$ also holds, because for $\alpha > \omega$ we still have that $x \in L_\alpha$. Seeing that $\forall\alpha\forall w(w \neq L_{\max(\omega,\alpha+4)} \vee \neg R(x, y, \alpha, w))$ also holds is not hard too, since $a > \omega$ implies that $L_{\max(\omega,\alpha+4)} = L_{\alpha+4}$ and $w = L_{\alpha+4} \to \neg R(y, x, \alpha, w)$. This proves that the bounded formula is equivalent to the unbounded one.

So we see that $\gamma = \max(\omega, \alpha + 5)$ suffices as an upper bound.

Exercise 2

Show that the formula $\mathrm{Enum}(\alpha, x)$ as is shown in [1, Lemma 3.6] fulfils the prerequisites of that lemma. That is, show that it is absolute for L and argue why the main statement holds in **KP**.

Solution to exercise 2 (3 pt)

First we prove the absoluteness result. From the fact that $\mathrm{Enum}(\alpha, x)$ is a Σ_1-formula, we immediately have that it is U-absolute. The result of D-absoluteness follows from the equivalence $\mathrm{Enum}(\alpha, x) \leftrightarrow \forall z(\mathrm{Enum}(\alpha, z) \to z = x)$.

There have been several solutions which have used different arguements. One can, for example, explicitly argue why both the free quantifiers in $\mathrm{Enum}(\alpha, x)$ can be bounded by L. This arguement, when done correctly, was awarded full points. Another arguement showed by induction on α that the function F could be constructed from its previous values. This arguement was also awarded full points.

The core of the second statement is that **KP** is strong enough to see that the function F is good enough and has the required properties. The fact that **KP** is strong enough follows easily from the fact that **KP** has Σ_1-collection and that F has the required properties is not hard to see, since that is pretty much the way that F is constructed.

Exercise 3

In this exercise we assume $V = L$. For each of the following statements, determine whether or not they are true (and explain why).

(i) A set X is finite[1] if and only if every injection $X \to X$ is also a surjection.

(ii) There is infinite κ such that $\kappa^\kappa \neq \kappa^+$.

(iii) The first uncountable cardinal ω_1 is singular.

Solution to exercise 3 (3 pt)

Each of the three parts is worth 1 point. The trick throughout the entire exercise was to realize that assuming $V = L$ we have both the Axiom of Choice (AC) and the Generalised Continuum Hypothesis (GCH).

(i) This is the notion of Dedekind finite, and assuming AC this is equivalent to finiteness. So this is true.

In more detail: let X be finite, then any injection $X \to X$ can be turned into an injection $n \to n$ for some natural n. An easy argument by induction then shows that any injection $n \to n$ must be surjective, which means that the original map $X \to X$ has to be surjective. So this part is already true without AC. For the converse we do need AC, and we will prove the contraposition. That is, if X is infinite then there is an injection $X \to X$ that is not a surjection. Since X is infinite we can (using AC!) find a subset Y of size ω in X. That is, $Y = \{y_0, y_1, \ldots\}$. Then we can define a function $X \to X$ by

$$x \mapsto \begin{cases} y_{n+1} & \text{if } x = y_n \\ x & \text{else} \end{cases}$$

which is injective but not surjective (as y_0 is not in its image).

[1] Recall that we defined a set X to be finite if there is a bijection $n \to X$ for some natural number n.

(ii) This is false. Using some simple cardinal arithmetic we find that $\kappa^\kappa = 2^\kappa$ (we do not need GCH here) for infinite κ, for example: apply [1, Lemma 5.2]. Since we have GCH, we have that for all infinite κ

$$\kappa^\kappa = 2^\kappa = \kappa^+.$$

(iii) This is again false, this time because of AC. For example [1, Lemma 5.7] tells us that every successor cardinal is regular.

It may be interesting to know that every one of these examples really needs either AC or GCH. That is, for each of these three statements its negation is consistent with **ZF** .

References

[1] Keith J. Devlin, *Constructibility*, Springer-Verlag Berlin, ISBN 0-387-13258-9, 1984.