The consistency of ZF + LO + $\neg$AC

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January 25, 2016

In [1], Bell constructs a Boolean-valued version of what is known as the *basic Cohen model* or the *first Cohen model* of ZF. Bell shows that in this model, denoted by $V^{(\Gamma)}$, there is a set s such that

$$V^{(\Gamma)} \models s \text{ is infinite, but not Dedekind infinite,}$$

thereby showing that the Axiom of Choice (AC) fails in this model.

Interestingly enough, one might say that this model does contain *some* choice. Indeed, in [2] it is shown that the statement 'every set can be linearly ordered' (LO) does hold in the model. By the equivalence of AC to the well-ordering theorem, it is easily seen that AC implies LO. This shows that LO is strictly weaker than AC. Since there are also models of ZF in which it does fail, *e.g.* in the *second Cohen model* (see [2]), it follows that ZF + LO is strictly in between ZF and ZF + AC in terms of strength.

In this small article we will prove that LO holds in the Boolean-valued *basic Cohen model*, as constructed by Bell in [1]. It should be read as an amateur addendum to Chapter 3, as we will use definitions, theorems and even variable names from Bell's book. The proof method is based on the analogous proofs in Chapter 4 and Chapter 5 of [2]. Let us begin with a definition.

Definition. Let $x \in V^{(\Gamma)}$. Then for finite $J \subseteq \omega$, we say that J is a *support* of x whenever $gx = x$ for every $g \in G_J$. ∎

It is clear that every $x \in V^{(\Gamma)}$ has a support. For, by definition, we have that $\mathrm{stab}(x) \in \Gamma$ and thus there is a finite $J \subseteq \omega$ such that $\mathrm{stab}(x) \subseteq G_J$. However, to prove that LO holds in $V^{(\Gamma)}$, we need to strengthen this result a bit.

Lemma. *Every $x \in V^{\Gamma}$ has a least support.*

Proof. We will prove this by showing that the intersection of all supports of x is itself a support of x.

First we will show that it holds for any two supports of x, and thus for any finite number. Let $J_1, J_2 \subseteq \omega$ be supports of x and let $J = J_1 \cap J_2$. In order to see that J is also a support of x, let $g \in G_J$. We will show that g can be written as a composition of permutations in $G_{J_1} \cup G_{J_2}$, and thus that $gx = x$. Notice that all permutations in G_J, G_{J_1}, G_{J_2} fix the elements in J, so we can assume wlog that $J = \emptyset$. We write $J_1 = \{x_1, \ldots, x_n\}$ and pick distinct $a_1, \ldots, a_n \in \omega \setminus (J_1 \cup J_2 \cup g(J_1))$. Consider the permutations

- $s \in G_{J_2}$ that swaps x_i for a_i for each i.
- $t_1 \in G_{J_2}$ that swaps a_i for $g(x_i)$ for each i s.t. $g(x_i) \in J_1$.
- $t_2 \in G_{J_1}$ that swaps a_i for $g(x_i)$ for each i s.t. $g(x_i) \notin J_1$.

Now consider the permutation $h = g^{-1}t_1t_2s$. Let $x_i \in J_1$, we claim that h fixes x_i and thus $h \in G_{J_1}$. Indeed,

- If $g(x_i) \in J_1$, we have $x_i \xrightarrow{s} a_i \xrightarrow{t_1} g(x_i) \xrightarrow{t_2} g(x_i) \xrightarrow{g^{-1}} x_i$.
- If $g(x_i) \notin J_1$, we have $x_i \xrightarrow{s} a_i \xrightarrow{t_1} a_i \xrightarrow{t_2} g(x_i) \xrightarrow{g^{-1}} x_i$.

And thus we can write $g = ghh^{-1} = gg^{-1}t_1t_2sh^{-1} = t_1t_2sh^{-1}$, to obtain the required expression of g.

Exercise 1 (1 pt.)**.** *Finish the above prove by extending this result to collections of supports of x of arbitrary length.*

□

Now that we have gotten this out of the way, we are ready to prove our theorem.

Theorem. $V^{(\Gamma)} \models$ *every set can be linearly ordered.*

Proof. We will prove this theorem by constructing for each set $u \in V^{(\Gamma)}$ a map $F : \mathrm{dom}(u) \to \mathrm{ORD} \times I$ in $V^{(\Gamma)}$ such that $V^{(\Gamma)} \models F$ is one-one, where I is the set of finite subsets of s in $V^{(\Gamma)}$.[1] As a theorem of ZF, we have that ORD can be linearly ordered in $V^{(\Gamma)}$ by the membership relation.

Exercise 2 (2 pt.)**.** *Show that the set I can also be linearly ordered in $V^{(\Gamma)}$. Hint: recall that $V^{(\Gamma)} \models s \subseteq P\hat{w}$ and use the axioms of ZF. You may also use the fact that Theorem 1.23 holds in $V^{(\Gamma)}$.*

Since every set $u \in V^{(\Gamma)}$ is the domain of another element in $V^{(\Gamma)}$ (e.g. of $\{\langle u, 1\rangle\}$ it will follow that $V^{(\Gamma)}$ proves that every set can be linearly ordered under the induced lexicographical ordering.

The idea is to associate with each $x \in \mathrm{dom}(u)$ a finite subset of s (corresponding to the least support of x) and an element of the set $O_u := \{\mathrm{orb}(x) \mid x \in \mathrm{dom}(u)\} \times \{1\}$, where $\mathrm{orb}(x) := \{gx \mid g \in G\} \times \{1\}$ is the *orbit* of x in $V^{(\Gamma)}$. It can be easily shown that $\mathrm{orb}(x), O_u \in V^{(\Gamma)}$ for $x, u \in V^{(\Gamma)}$.

Let us first show that O_u can be well-orderd in $V^{(\Gamma)}$ and thus that there is a one-to-one function from O_u into ORD. Analogously to the definition of an *ordered pair* in $V^{(B)}$ on p. 52 of Bell, we define for $u, v \in V^{(\Gamma)}$

$$\begin{aligned} \{u\}^{(\Gamma)} &= \{\langle u, 1\rangle\}; \\ \{u, v\}^{(\Gamma)} &= \{u\}^{(\Gamma)} \cup \{v\}^{(\Gamma)}; \\ \langle u, v\rangle^{(\Gamma)} &= \{\{u\}^{(\Gamma)}, \{u, v\}^{(\Gamma)}\}^{(\Gamma)}. \end{aligned}$$

Exercise 3 (1 pt.)**.** *Prove that $\{u\}^{(\Gamma)}, \{u, v\}^{(\Gamma)}, \langle u, v\rangle^{(\Gamma)} \in V^{(\Gamma)}$ for all $u, v \in V^{(\Gamma)}$.*

Now define $f := \{\langle \hat{z}, z\rangle^{(\Gamma)} \mid z \in \mathrm{dom}(O_u)\} \times \{1\}$. By the previous exercise and Lemma 3.14, we have that $\mathrm{dom}(f) \subseteq V^{(\Gamma)}$. Furthermore, it holds for $g \in G$, that $gz = g\mathrm{orb}(x) = \mathrm{orb}(gx) = \mathrm{orb}(x) = z$ and $g\hat{z} = \hat{z}$ and thus

$$\mathrm{stab}(f) = \{g \in G \mid gf = f\} = G \in \Gamma.$$

By the well-ordering theorem in V, we can pick an ordinal α and a bijection g of α onto $\mathrm{dom}(u)$. It follows that

$$V^{(\Gamma)} \models \hat{g} \text{ is a bijection of the ordinal } \hat{\alpha} \text{ onto } \widehat{\mathrm{dom}(O_u)}.$$

Consequently, we have

$$V^{(\Gamma)} \models f \circ \hat{g} \text{ is a function with domain } \hat{\alpha} \text{ and range extending } O_u,$$

and thus $V^{(\Gamma)}$ proves that O_u is well-orderable.

Exercise 4 (1 pt.)**.** *It is a theorem of ZF that every infinite well-ordered set is Dedekind infinite. We have seen in [1] that while it is infinite, the set s is not Dedekind infinite. Show that the above method indeed fails when one tries to use it to prove the well-orderedness of s, by showing that $f' := \{\langle \hat{z}, z\rangle^{(\Gamma)} \mid z \in \mathrm{dom}(s)\} \times \{1\} \notin V^{(\Gamma)}$.*

[1] In other versions of this proof, the function F is constructed as a class function that has the whole universe as its domain. In our construction this would probably also work, but the nice thing about this way is that we now have $F \in V^{(\Gamma)}$.

We construct our map $F : \mathrm{dom}(u) \to \text{ ORD} \times I$ in two parts. First define the function

$$F_1 := \{\langle x, \mathrm{orb}(x)\rangle^{(\Gamma)} \mid x \in \mathrm{dom}(u)\} \times \{1\}.$$

Again, it can be readily verified that $F_1 \in V^{(\Gamma)}$. Secondly, we define

$$F_2 := \{\langle x, \{u_{n_1}, \ldots, u_{n_j}\} \times \{1\}\rangle^{(\Gamma)} \mid x \in \mathrm{dom}(u)\} \times \{1\},$$

where $\{n_1, \ldots, n_j\}$ is the least support of x. Once again, the straightforward verification that $F_2 \in V^{(\Gamma)}$ is left to the reader.

To see that the function obtained from combining F_1 and F_2 in $V^{(\Gamma)}$ is one-one, suppose that for $x, y \in V^{(\Gamma)}$, we have

$$V^{(\Gamma)} \models F_1(x) = F_1(y) \wedge F_2(x) = F_2(y).$$

We claim that then $V^{(\Gamma)} \models x = y$. Firstly, since we know that $V^{(\Gamma)} \models u_n \neq u_{n'}$ for $n \neq n'$, it holds that x and y have the same least support J. Furthermore, notice that

$$\begin{aligned} 1 &= [\![\mathrm{orb}(x) = \mathrm{orb}(y)]\!] \\ &\leq \bigwedge_{g \in G} [\![gx \in \mathrm{orb}(y)]\!] \\ &\leq [\![x \in \mathrm{orb}(y)]\!] = \bigvee_{g \in G} [\![x = gy]\!], \end{aligned}$$

This means that $\bigwedge_{g\in G} [\![x \neq gy]\!] = 0$ and thus that for all $p \in P$ we have $p \not\leq \bigwedge_{g\in G} [\![x \neq gy]\!]$. That is, there is a $g \in G$ such that $p \not\leq [\![x \neq gy]\!]$ or such that $p \not\Vdash x \neq y$. It follows that for that g, there is a $q \in P$ with $q \leq p$ such that

$$q \Vdash x = gy.$$

Now suppose, by contradiction, that $V^{(\Gamma)} \not\models x = y$. Then there is some $p \in P$ such that $p \Vdash x \neq y$. By the above, there is a $q \leq p$ and $g \in G$ such that $q \Vdash x = gy$.

Therefore, we have that $q \Vdash F_2(y) = F_2(x) = F_2(gy)$. A simple calculation of Boolean truth value shows that this must mean that gy also has least support J, for otherwise we would have $[\![F_2(y) = F_2(gy)]\!] = 0 \not\geq q$.

It can easily be verified that since J is a support of y, that gJ is a support of gy. Since J and gJ have the same size, and J is the least support of y, we must have $J = gJ$ and thus $g \in G_J$. But then $q \Vdash x = gy = y$, which is a contradiction since $q \leq p$. $\square$

References

[1] John L Bell. *Set theory: Boolean-valued models and independence proofs.* Oxford University Press, 2005.

[2] Thomas Jech. *The Axiom of Choice.* 1973.