Seminar on Set Theory

Handout 3

October 2, 2015

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1 Induction principle for $V^{(B)}$

We can prove that for any formula $\phi(x)$,

$$\forall x \in V^{(B)}[\forall y \in \mathrm{dom}(x)\phi(y) \to \phi(x)] \to \forall x \in V^{(B)}\phi(x).$$

2 Definition of truth value map $[\![\cdot]\!]^B$

Let σ and τ be B-sentences. Define

$$\begin{aligned} [\![\sigma \wedge \tau]\!]^B &= [\![\sigma]\!]^B \wedge [\![\tau]\!]^B; \\ [\![\neg\sigma]\!]^B &= ([\![\sigma]\!]^B)^*. \end{aligned}$$

If $\phi(x)$ is a B-formula with one free variable x such that $[\![\phi(u)]\!]^B$ has been defined for all $u \in V^{(B)}$, define

$$\begin{aligned} [\![\exists x\phi(x)]\!]^B &= \bigvee_{u \in V^{(B)}} [\![\phi(u)]\!]^B; \\ [\![\sigma \vee \tau]\!]^B &= [\![\sigma]\!]^B \vee [\![\tau]\!]^B; \\ [\![\sigma \to \tau]\!]^B &= [\![\sigma]\!]^B \Rightarrow [\![\tau]\!]^B; \\ [\![\forall x\phi(x)]\!]^B &= \bigwedge_{u \in V^{(B)}} [\![\phi(u)]\!]^B. \end{aligned}$$

Defining the Boolean truth values of the atomic formulas ($u = v$, $u \in v$) is more difficult. Rewrite:

$$\begin{aligned} [\![u = v]\!]^B &= [\![\forall x \in u[x \in v] \wedge \forall y \in v[y \in u]]\!]^B; \\ [\![u \in v]\!]^B &= [\![\exists y \in v[u = y]]\!]^B. \end{aligned}$$

We require that formules with bounded quantifiers have the form

$$\begin{aligned} [\![\exists x \in u\phi(x)]\!]^B &= \bigvee_{x \in dom(u)} [u(x) \wedge [\![\phi(x)]\!]^B]; \\ [\![\forall x \in u\phi(x)]\!]^B &= \bigwedge_{x \in dom(u)} [u(x) \Rightarrow [\![\phi(x)]\!]^B]. \end{aligned}$$

We use the equations above and recursion on a well-founded relation to use as a definition

$$\begin{aligned} [\![u \in v]\!]^B &= \bigvee_{y \in dom(v)} [v(y) \wedge [\![u = y]\!]^B]; \\ [\![u = v]\!]^B &= \bigwedge_{x \in dom(u)} [u(x) \Rightarrow [\![x \in v]\!]^B] \\ &\quad \wedge \bigwedge_{y \in dom(v)} [v(y) \Rightarrow [\![y \in u]\!]^B]. \end{aligned}$$

Recursion on well-founded relations

The principle of recursion on a well-founded relation R is the assertion that if F is any class of ordered pairs, which defines a single-valued mapping of V into V (such a class is called a function on V), then there is a function G on V such that

$$\forall u[G(u) = F(\langle u, G|Ru \rangle)].$$

$V^{(B)}$ respects the axioms of first-order logic

Theorem 2.1. *The axioms and rules of inference of first-order logic hold in $V^{(B)}$. In particular, we have for all $u, v, w \in V^{(B)}$,*

1. $[\![u = u]\!] = 1$;
2. $u(x) \leq [\![x \in u]\!]$ *for* $x \in dom(u)$;
3. $[\![u = v]\!] = [\![v = u]\!]$;
4. $[\![u = v]\!] \wedge [\![v = w]\!] \leq [\![u = w]\!]$.

$V^{(B)}$ can be used to prove relative consistency to ZF

Theorem 2.2. *Let T, T' be extensions of* ZF *such that $Consis(\mathrm{ZF}) \to Consis(T')$, and suppose that in $\mathcal{L}$ we can define a constant term B, such that*

$$T' \vdash \text{"}B \text{ is a Boolean algbera"},$$

and for each axiom $\tau \in T$, we have

$$T' \vdash [\![\tau]\!]^B = 1_B.$$

Then $Consis(\mathrm{ZF}) \to Consis(T)$.