Seminar on Set Theory

Hand-out lecture 7

November 6, 2015

Part I - Refined posets and Boolean completions

Definition 1. *For* $u, v \in V^{(B)}$ *define* $\{u\}^{(B)} = \{\langle u, 1\rangle\}$, $\{u, v\}^{(B)} = \{u\}^{(B)} \cup \{v\}^{(B)}$ *and* $\langle u, v\rangle^{(B)} = \{\{u\}^{(B)}, \{u, v\}^{(B)}\}^{(B)}$.

Theorem 2. *For all* $u, v \in V^{(B)}$ *there exists an* $f \in V^{(B)}$ *such that*

$$V^{(B)} \models \mathrm{Fun}(f) \wedge \mathrm{dom}(f) = \widehat{\mathrm{dom}(u)} \wedge u \subseteq \mathrm{ran}(f).$$

Definition 3. *Let* $(P, \leq)$ *be a poset. Elements* $p, q \in P$ *are called* compatible*, written* $\mathrm{Comp}(p, q)$*, if* $\exists r \in P(r \leq p \wedge r \leq q)$*, and* $(P, \leq)$ *is called* refined *if*

$$\forall p, q \in P(q \not\leq p \rightarrow \exists p' \leq q \neg \mathrm{Comp}(p, p')).$$

The sets $O_p = \{q \in P \ : \ q \leq p\}$ form a basis for the *left order topology* on P. A set $X \subseteq P$ is open in this topology iff $\forall p, q((p \leq q \wedge q \in X) \rightarrow p \in X)$.

Theorem 4. P *is refined iff* $O_p \in \mathrm{RO}(P)$ *for all* $p \in P$.

Definition 5. $X \subseteq B$ *is called* dense *in* B *if* $0 \notin X$ *and for all* $0 \neq b \in B$ *there exists an* $x \in X$ *such that* $x \leq b$.

Theorem 6. *P is refined iff it is order-isomorphic to a dense subset of a complete Boolean algebra.*

Definition 7. *A pair* $\langle B, e\rangle$ *is called a* Boolean completion *of* P *if* B *is a complete Boolean algebra and* e *is an order-isomorphism of* P *onto a dense subset of* B.

Theorem 8. *Boolean completions are unique up to isomorphism of Boolean algebra's.*

Part II - Forcing and the consistency of $V \neq L$

Definition. Let P be a refined poset an let (B, e) be a Boolean completion of P. We identify the image $e(P) \subseteq B$ with P itself. For $p \in P$ and a B-sentence σ, we define the relation $p \Vdash \sigma$ by

$$p \Vdash \sigma \qquad \text{iff} \qquad p \leq [\![\sigma]\!]^B.$$

We say that p *forces* σ.

Properties. Let σ and τ be B-sentences. Then:

(i) For all $p, q \in P$, we have: if $q \leq p$ and $p \Vdash \sigma$, then $q \Vdash \sigma$.

(ii) If $[\![\sigma]\!]^B = 1$, then $p \Vdash \sigma$ for all $p \in P$.

(iii) For all $p \in P$, we have: $p \Vdash \sigma \wedge \tau$ if and only if $p \Vdash \sigma$ and $p \Vdash \tau$.

(iv) $[\![\sigma]\!]^B = 0$ if and only if there are no $p \in P$ such that $p \Vdash \sigma$.

(v) For all $p \in P$, we have: $p \Vdash \neg\sigma$ if and only if $\neg\exists q \leq p \ (q \Vdash \sigma)$.

(vi) For all $p \in P$, we have: if $p \Vdash \neg\sigma$, then $p \nVdash \sigma$. Equivalently, if $p \Vdash \sigma$, then $p \nVdash \neg\sigma$.

(vii) For all $p \in P$, we have: $p \Vdash \sigma \to \tau$ if and only if $\forall q \leq p \ (q \Vdash \sigma \to q \Vdash \tau)$.

Theorem 1. (Bell 2.6) Let $B = \mathrm{RO}(2^\omega)$. Then

- $V^{(B)} \models \widehat{\mathcal{P}\omega} \neq \mathcal{P}\hat{\omega}$.
- $V^{(B)} \models \mathcal{P}\hat{\omega} \nsubseteq L$.

Theorem 2. (Bell 1.19) Let T and T' be extensions of ZF such that Consis(ZF) $\to$ Consis(T'), and suppose that in the language of set theory, we can define a constant term B such that

$$T' \vdash B \text{ is a complete Boolean algebra and, for each axiom } \tau \text{ of } T, \text{ we have } T' \vdash [\![\tau]\!]^B = 1_B.$$

Then Consis(ZF) $\to$ Consis(T).

Corollary 3. (Bell 2.7) If ZF is consistent, then so is ZFC $+$ $(\mathcal{P}\omega \nsubseteq L)$.

Theorem 4. (Bell 2.8) Suppose the GCH holds and that B is a complete Boolean algebra satisfying the ccc and $|B| = 2^{\aleph_0}$. Then

$$V^{(B)} \models \mathrm{GCH}.$$

Corollary 5. (Bell 2.9) If ZF is consistent, then so is ZFC $+$ GCH $+$ $(\mathcal{P}\omega \nsubseteq L)$.