# Seminar on Set Theory

Hand-in exercise 1, model solution

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(a) Let $x, y \in H$ such that $x \leq y$. We know that $y \leq y^{**}$ so by transitivity of $\leq$ we find that $x \leq y^{**}$, and this is equivalent to $y^* \leq x^*$. $\square$

*Alternative.* If $x \leq y$, then $y = x \vee_H y$. Taking complements, we find that $y^* = (x \vee_H y)^* = x^* \wedge_H y^*$, which means exactly that $y^* \leq x^*$, $\square$

(b) First of all, notice that $x^* \in B$ for all $x \in H$, since we have $x^{***} = x^*$. Also, using exercise (a) we see that $x \leq y$ implies $y^* \leq x^*$, which in turn implies $x^{**} \leq y^{**}$. So the map $H \to H : x \mapsto x^{**}$ preserves order.

Since $0_H \leq 0_H$, we have $0_H^* = (0_H \Rightarrow 0_H) = 1_H$. Also, there is obviously only one $z \in H$ such that $z \wedge_H 1_H = 0_H$, namely $z = 0_H$. This means that $1_H^* = (1_H \Rightarrow 0_H) = 0_H$. From these facts we deduce that $0_H, 1_H \in B$, so $B$ has a greatest and a least element, and these are induced from $H$.

Let us show that for all $x, y \in B$, we have $x \wedge_H y \in B$. Since $x \wedge_H y \leq x$, we have $(x \wedge_H y)^{**} \leq x^{**} = x$. Similarly, we have $(x \wedge_H y)^{**} \leq y$. From these it follows that $(x \wedge_H y)^{**} \leq x \wedge_H y$. But we also have $x \wedge_H y \leq (x \wedge_H y)^{**}$, so it is indeed the case that $x \wedge_H y \in B$. Now clearly, any $z \in B$ that is a lower bound of $x$ and $y$ must satisfy $z \leq x \wedge_H y$. But the latter is itself in $B$, so we can take $x \wedge_H y$ to be the infimum of $x$ and $y$ in $B$.

Again, let $x, y \in B$ be given. Clearly, any $z \in B$ that is an upper bound of $x$ and $y$ must satisfy $z \geq x \vee_H y$. From this it follows that $z = z^{**} \geq (x \vee_H y)^{**}$. But $(x \vee_H y)^{**}$, being the pseudocomplement of something, is in $B$. So we can take $(x \vee_H y)^{**}$ to be the supremum of $x$ and $y$ in $B$. We conclude that $B$ is a bounded lattice.

Finally, we have $x^* \in B$ for all $x \in B \subset H$, as we already remarked. We have $x \wedge_B x^* = x \wedge_H x^* = 0_H = 0_B$ and $x \vee_B x^* = (x \vee_H x^*)^{**} = (x^* \wedge_H x^{**})^* = 0_H^* = 1_H = 1_B$. So $B$ is a complemented bounded lattice, i.e. a Boolean algebra. $\square$

(c) Suppose that $H$ is complete and let $X \subset B$. Then $X$ has a supremum $\bigvee X$ in $(H, \leq)$. Now every $z \in B$ that is an upper bound of $X$ must certainly satisfy $z \geq \bigvee X$. From this it follows that $z = z^{**} \geq (\bigvee X)^{**}$. But $(\bigvee X)^{**}$, being the pseudocomplement of something, is itself in $B$. So we can take $(\bigvee X)^{**}$ to be the supremum of $X$ in $B$. The existence of infima can be shown similarly. $\square$

(d) We have to prove that

$$\overset{\circ}{\overline{U}} = U \text{ iff } U = \overset{\circ}{\widehat{X - \overset{\circ}{\widehat{X - U}}}}.$$

We will do this by proving that

$$\overline{U} = X - \overset{\circ}{\widehat{X - U}}$$

and using the fact from topology that if $A \subset B$ then $\mathring{A} \subset \mathring{B}$. We notice that

$$\begin{aligned} a \in \left( X - \widehat{X - U}^{\circ} \right) \quad &\text{iff} \quad \neg \left( a \in \widehat{X - U}^{\circ} \right) \\ &\text{iff} \quad \neg \exists \delta > 0 (B(a;\delta) \subset X - U) \\ &\text{iff} \quad \forall \delta > 0 (B(a;\delta) \cap U \neq \emptyset) \\ &\text{iff} \quad a \in \overline{U} \end{aligned}$$

And therefore $RO(X)$ is the regularization of $O(X)$. □

(e) We use the example from (d) to show this. Suppose $X = \mathbb{R}$, equipped with the Euclidean topology. Let $U = (1,2)$ and $V = (2,3)$. Then in $O(X)$ the meet of these opens is just $(1,2) \cup (2,3)$, which does not contain the point 2. However, the meet in the regularization looks as follows. First we take the complement in $\mathbb{R}$ of $(1,2) \cup (2,3)$, which is $(-\infty, 1] \cup \{2\} \cup [3, \infty)$. The interior of this is $(-\infty, 1) \cup (3, \infty)$, which has $[1,3]$ as complement. The interior of this is $(1,3)$, and that is the meet of $U$ and $V$ in the regularization of $O(\mathbb{R})$. It follows that the meet in the regularization of $O(\mathbb{R})$ is not induced from the meet in $O(\mathbb{R})$. □