Seminar on Set Theory

Hand-in exercise 15

January 15, 2016

Exercise 1.

(i) First we show that Collection implies Replacement. If $\forall x \in u \exists ! y \varphi(x,y)$ is the case, then by Collection we can form a set v such that $\forall x \in u \exists y \in v \varphi(x,y)$. We apply Separation on v to find the set $w = \{y \in v : \exists x \in u \varphi(x,y)\}$. Clearly if $y \in w$ then $\exists x \in u \varphi(x,y)$; on the other hand, if for some y we have $\exists x \in u \varphi(x,y)$, then because of uniqueness of y we must have $y \in v$ and so $y \in w$.
Now to show that Replacement implies Collection. Suppose that $\forall x \in u \exists y \varphi(x,y)$. Define $\psi(x,w)$ by $\exists \alpha \in \mathrm{ORD}[w = V_\alpha \wedge \forall \beta \in \alpha[\alpha = \beta \leftrightarrow \exists y \in V_\beta \varphi(x,y)]]$. By Regularity we have $V = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$, hence $\forall x \in u \exists ! w \psi(x,w)$. By Replacement, there is some u such that $\forall w[w \in u \leftrightarrow \exists x \in u \psi(x,w)]$. By Union we can form $\bigcup u$, of which we recognize that $\forall x \in u \exists y \in \bigcup u \varphi(x,y)$, which concludes the proof. □
Points awarded: $\frac{1}{2}$ point per correctly proven direction of the equivalence, where the solution is required to explicitly mention the crucial axioms.

(ii) First we show that Set Induction implies Regularity. From Set Induction we can conclude that $V = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$, so we can use the notion of rank of a set. Thus if $u \neq \emptyset$ is any set, then since the ordinals are well-founded it must have an element $x \in u$ of lowest rank, for which it must hold that $x \cap u = \emptyset$ as required. [There are more direct and elementary proofs as well.]
Now to show that Regularity implies Set Induction. Suppose $\forall x[\forall y \in x \varphi(y) \to \varphi(x)]$, so that for any x we have $\neg\varphi(x) \to x^- \neq \emptyset$, where x^- denotes the set $\{y \in x : \neg\varphi(y)\}$. Assume we have some set x such that $\neg\varphi(x)$. Then there is some $z_1 \in x^-$: but then since $\neg\varphi(z_1)$ again holds, there is in turn some $z_2 \in z_1^-$, which by iteration results in an infinite sequence $x \ni z_1 \ni z_2 \ni \ldots$. By Replacement these form a set which contradicts Regularity, so we must have $\forall x \varphi(x)$, which completes the proof. □
Points awarded: 1 point per correctly proven direction of the equivalence, where the solution is required to explicitly mention the crucial axioms.

(iii) First we show that Fullness implies Subset Collection. For u and v, let $F_{u,v}$ be a full set of total relations between u and v as given by Fullness. For any relation R between u and v we can form by Restricted Separation the set $v_R = \{y \in v : \exists x \in u \langle x,y\rangle \in R\}$. By Strong Collection there is a set $W_{u,v} = \{v_R : R \in F_{u,v}\}$. Now let z be arbitrary, and suppose that $\forall x \in u \exists y \in v \varphi(x,y,z)$. We would like some total relation R_z between u and v which is based on z. We cannot straightforwardly define R_z as $\{\langle x,y\rangle : x \in u \wedge y \in v \wedge \varphi(x,y,z)\}$ since $\varphi(x,y,z)$ is generally not restricted. Therefore we instead obtain R_z via Strong Collection on the formula $\psi(x,p)$ which is $\exists y \in v[p = \langle x,y\rangle \wedge \varphi(x,y,z)]$. We see that R_z thus defined is indeed a total relation between u and v, and that $\varphi(x,y,z)$ is the case whenever $\langle x,y\rangle \in R_z$. Now there is some $R \in F_{u,v}$ such that $R \subseteq R_z$, along with corresponding $v_R \in W_{u,v}$. Since R is again total, for all $x \in u$ there is some $y \in v_R$ such that $\langle x,y\rangle \in R$, but then also $\langle x,y\rangle \in R_z$ and so $\varphi(x,y,z)$ holds. On the other hand, if $y \in v_R$, then by definition there is some $x \in u$ such that $\langle x,y\rangle \in R$ and so $\langle x,y\rangle \in R_z$, by which we again obtain $\varphi(x,y,z)$. This shows that Subset Collection holds with $W_{u,v}$ as witness.

Now to show that Subset Collection implies Fullness. For u and v, consider the formula $\varphi(x,p,z)$ given by $p \in z \wedge \exists y \in v(p = \langle x,y \rangle)$. By Subset Collection we find a set $w_{u,u\times v}$ such that for any z we have $\forall x \exists p \in u \times v \varphi(x,p,z) \rightarrow \exists P_z \in w_{u,u\times v}[\forall x \in u \exists p \in P_z \varphi(x,p,z) \wedge \forall p \in P_z \exists x \in u \varphi(x,p,z)]$. Now let z be any total relation between u and v, so that for every $x \in u$ there is some $y \in v$ such that $\langle x,y \rangle \in z$. Then clearly $\forall x \exists p \in u \times v \varphi(x,p,z)$ is the case, hence there is some $P_z \in w_{u,u\times v}$ such that $\forall x \in u \exists p \in P_z \varphi(x,p,z)$ as well as $\forall p \in P_z \exists x \in u \varphi(x,p,z)$. The latter tells us that $P_z \subseteq z$ is a relation between u and v, and the former gives us moreover that P_z is total, thus for any total relation z the set $w_{u,u\times v}$ contains a total relation P_z which refines it. Since being a total relation is restricted, by Restricted Separation we may consider $f_{u,u\times v} = \{R \in w_{u,u\times v} : \mathrm{TRel}(R,u,v)\}$, which then witnesses Fullness for our arbitrary u and v. □

Points awarded: 1 point per correctly proven direction of the equivalence, where the solution is required to explicitly mention the crucial axioms.

Exercise 2.

(i) If $\alpha < \beta^+$, then $\alpha \in \beta \cup \{\beta\}$, so $\alpha \in \beta$ or $\alpha = \beta$. In the first case, the transitivity of β yields that $\alpha \subset \beta$. In the second case, we also have $\alpha \subset \beta$. We conclude that $\alpha \leq \beta$. □

(ii) Suppose that $\forall \alpha, \beta\ (\alpha \leq \beta \rightarrow \alpha < \beta^+)$ holds. We let $\alpha = 0$ and $\beta = \{0 \mid \phi\}$. Then clearly $0 \subset \{0 \mid \phi\}$, so we get $0 \in \{0 \mid \phi\}^+$. That is, $0 \in \{0 \mid \phi\}$ or $0 = \{0 \mid \phi\}$. In the first case, it follows that ϕ, while in the second case, it follows that $\neg\phi$. □
Alternatively, one may take $\alpha = \{0 \mid \phi\}$ and $\beta = 1$.

(iii) Suppose that $\forall \alpha, \beta, \gamma\ (\alpha \leq \beta < \gamma \rightarrow \alpha < \gamma)$ holds. We let $\gamma = \beta^+$. Since $\beta < \beta^+$ always holds, we now get $\forall \alpha, \beta\ (\alpha \leq \beta \rightarrow \alpha < \beta^+)$, which implies LEM, by exercise (ii). □

(iv) We apply this to $\alpha = \{0 \mid \phi\}$. Suppose that $\{0 \mid \phi\}$ is a weak limit. Suppose we have a $\beta \in \{0 \mid \phi\}$, then there must also be a $\gamma \in \{0 \mid \phi\}$ such that $\beta \in \gamma$. But since β and γ are both in $\{0 \mid \phi\}$, we have $\beta = \gamma = 0$. However, we clearly do not have $0 \in 0$, contradiction. This shows that $\neg\exists\beta \in \{0 \mid \phi\}$, so $\{0 \mid \phi\} = 0$, whence $\neg\phi$ holds. If $\{0 \mid \phi\} = \beta^+$ for some ordinal β, then we have $\beta \in \{0 \mid \phi\}$, so $\beta = 0$. This means that $\{0 \mid \phi\} = 0^+ = \{0\}$, so ϕ holds. □
One point was awarded for correctly handling the case in which $\{0 \mid \phi\}$ is a successor, and one point for the case in which it is a weak limit.