Seminar on Set Theory

Solutions to exercise 2

September 25, 2015

1 Boolean Algebras and Propositional Logic (6 points total)

Let $\mathcal{L}$ be the classical propositional language consisting of $\bot, \top, \neg, \to, \vee, \wedge, \leftrightarrow$ and propositional variables P_x for each $x \in B$. Let the theory T have the following sentences for all $x, y \in B$:

$$P_x \wedge P_y \leftrightarrow P_{x \wedge y}, \tag{1}$$
$$P_{x^*} \leftrightarrow \neg P_x, \tag{2}$$
$$\neg(P_x \to P_y) \quad \text{if } x \not\leq y. \tag{3}$$

(Students get $1,5$ *points for defining a good theory)*

Define a map $f \colon B \to B(T)$ by $x \mapsto [P_x]$ for all $x \in B$. We will show that f is a bijective algebra homomorphism.

If $x, y \in B$, then

$$\begin{aligned} f(x \wedge y) &= [P_{x \wedge y}] \\ &= [P_x \wedge P_y] && \text{(by (1))} \\ &= [P_x] \wedge [P_y] && \text{(by construction of } B(T)) \\ &= f(x) \wedge f(y). \end{aligned}$$

Furthermore,

$$\begin{aligned} f(x^*) &= [P_{x^*}] \\ &= [\neg P_x] && \text{(by (2))} \\ &= [P_x \to \bot] \\ &= [P_x] \to [\bot] && \text{(by construction of } B(T)) \\ &= [P_x]^* \\ &= f(x)^*. \end{aligned}$$

Proposition 1.1 now tells us that f is an algebra homomorphism. *(Students get* 1 *point for showing this)*

To show that f is injective, suppose we have $x, y \in B$ with $x \neq y$. We may assume that $x \not\leq y$. By (3), we must have that $T \vdash \neg(P_x \to P_y)$, so

$T \not\vdash P_x \to P_y$. Hence, by construction of $B(T)$, we have $f(x) = [P_x] \neq [P_y] = f(y)$. *(Students get 2 point for showing injectivity)*

We prove that f is surjective by induction on the complexity of formulas of $B(T)$. Clearly, for every propositional variable P_x of $\mathcal{L}$, there is an $x \in B$ such that $f(x) = [P_x]$. Furthermore, $f(0) = [\bot]$ and $f(1) = [\top]$ (by Proposition 1.1). Assume that there are $x, y \in B$ such that $f(x) = [\phi]$ and $f(y) = [\psi]$, then clearly:

$$\begin{aligned}
&f(x^*) = [\phi]^* = [\neg\phi],\\
&f(x \wedge y) = [\phi] \wedge [\psi] = [\phi \wedge \psi],\\
&f(x \vee y) = [\phi] \vee [\psi] = [\phi \vee \psi],\\
&f(x \Rightarrow y) = [\phi] \Rightarrow [\psi] = [\phi \to \psi],\\
&f((x \Rightarrow y) \wedge (y \Rightarrow x)) = [\phi \leftrightarrow \psi] \quad \text{(by the equalities above).}
\end{aligned}$$

(Students get 2 points for showing surjectivity) We conclude that f is a bijective algebra homomorphism, so B and $B(T)$ are isomorphic.

2 Cantor's Theorem (4 points total)

a) Suppose Cantor's Theorem is not true. Then there is a set X and a bijection $f : X \to \mathcal{P}(X)$ (It is trivial to show that the power set of a set is not of lower cardinality than that set). Clearly X is nonempty because $\emptyset$ and $\{\emptyset\}$ have different finite cardinality. Define the subset $X_0 \subseteq X$ by:

$$X_0 = \{x \in X : x \notin f(x)\}.$$

By Zermelo's third Axiom of separation this indeed is a set. And it is a subset of X, so $X_0 \in \mathcal{P}(X)$. Since f is a bijection, there must be a $x \in X$ such that $X_0 = f(x)$. Like in Zermelo's proof we again consider the two possible cases:

$x \in X_0$: Then, by definition of X_0, $x \notin f(x) = X_0$ which is a contradiction.

$x \in X \setminus X_0$: Then x is an element of X, not in $f(x)$. So $x \in X_0$ which again is a contradiction.

Thus, we get a contradiction and conclude that there is no such bijection. Thereby proving the theorem. *(Students get 3 points for Proving the theorem)*

b) Suppose such a set U does exist. Then by the powerset axiom $\mathcal{P}(U)$ is also a set and all its elements are contained in U. So $|\mathcal{P}(U)| \leq |U|$ which is in contradiction with Cantor's theorem. *(Students get 1 point for showing this)*