Hand-in 16: Seminar Set Theory

January 25, 2016

Problem 1: Category Theory

When defining products, exponential objects, and monomorphisms during the lecture, we first considered the construction in terms of morphisms $1 \to A$, and then generalised this to morphisms $Z \to A$ for any object Z.

For any one of these constructions:

(a, 0.5 points): Show that in Set, the definition remains equivalent if we replace "for any object Z" with "for some terminal object 1".

(b, 0.5 points): Show that this is not necessarily the case for an arbitrary topos. (Hint: Set^2 with all operations defined componentwise is a topos.)

Now let $\mathcal{C}$ be an arbitrary category with a terminal object 1.

(c, 0.5 point): Show that for any object A of $\mathcal{C}$, any morphism $x : 1 \to A$ is a monomorphism.

Assume furthermore that $\mathcal{C}$ is an elementary topos.

(d, 0.5 point): Let 0 be an object such that for all $Z \in Ob$, there exists exactly one morphism $f : 0 \to Z$. Show that f is a monomorphism.

Let $A \in Ob$ and supppose the product $A \times 1$ exists in $\mathcal{C}$. We say two objects A and B are isomorphic if there exist morphisms $f : A \to B$ and $g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$.

(e, 0.5 point): Show that A and $A \times 1$ are isomorphic.

Problem 2: Truth in Set_H

Let H be a complete Heyting algebra.

Definition 1. *An H-set is a pair (X, δ) where $\delta : X \times X \to H$ satisfying*

$$\begin{aligned} \delta(x, x') &= \delta(x', x) && \forall x, x' \in X \\ \delta(x, x') \wedge \delta(x', x'') &\leq \delta(x, x'') && \forall x, x', x'' \in X. \end{aligned}$$

Definition 2. *An H-valued functional relation (for brevity: H-function) from (X, δ_X) to (Y, δ_Y) is a function $f : X \times Y \to H$ satisfying*

$$\begin{aligned} \delta_X(x, x') \wedge f(x', y) &\leq f(x, y) && \forall x, x' \in X, y \in Y \\ f(x, y) \wedge \delta_Y(y, y') &\leq f(x, y') && \forall x \in X, y, y' \in Y \\ f(x, y) \wedge f(x, y') &\leq \delta_Y(y, y') && \forall x \in X, y, y' \in Y \\ \bigvee_{y \in Y} f(x, y) &= \delta_X(x, x) && \forall x \in X. \end{aligned}$$

Define composition of $f : (X, \delta_X) \to (Y, \delta_Y)$ and $g : (Y, \delta_Y) \to (Z, \delta_Z)$ by

$$(g \circ f)(x, z) = \bigvee_{y \in Y} \left(f(x, y) \wedge g(y, z) \right).$$

(2.5 points): We claim Set_H is a topos. Find the terminal object 1, the truth-value object Ω, and the $true : 1 \to \Omega$ morphism and show that they satisfy definitions 2 and 7 of the handout.