Seminar on Set Theory
Hand-in Exercise 5

Exercise 1

(i)[2 points] Without using the axiom of choice, prove that there is an ordinal α such that

$$\bigvee_{y \in V^{(B)}} [\![\phi(y)]\!] = \bigvee_{y \in V_\alpha^{(B)}} [\![\phi(y)]\!]$$

Hint: You only need to use the axiom of replacement on V and the fact that B is a set.

(ii)[1 point] Now prove (still without AC) that for any set $u \subseteq V^{(B)}$ there is an ordinal α such that for all $x \in u$:

$$\bigvee_{y \in V^{(B)}} [\![\phi(x,y)]\!] = \bigvee_{y \in V_\alpha^{(B)}} [\![\phi(x,y)]\!]$$

Exercise 2

Let $u \in V^{(B)}$ such that $u(x) = 1$ for all $x \in \mathrm{dom}(u)$, and define $w \in V^{(B)}$ by $w = B^{\mathrm{dom}(u)} \times \{1\}$.

(i) [1 point] Why does $[\![\forall x[x \in w \leftrightarrow x \subseteq u]]\!] = 1$ hold?

Now for $u \in V^{(B)}$ freely, define $w \in V^{(B)}$ by $w = B^{\mathrm{dom}(u)} \times \{1\}$.

(ii) [1 point] Give an example for u and B such that $[\![\forall x[x \in w \leftrightarrow x \subseteq u]]\!] \neq 1$.

Exercise 3

Consider $\langle X, \leq_X \rangle \in V^{(B)}$ such that the formula expressing that $\langle X, \leq_X \rangle$ is a nonempty inductive poset is true in $V^{(B)}$. Let Y be a core of X, and define the relation $\leq_Y$ on Y by $y \leq_Y y' \leftrightarrow [\![y \leq_X y']\!] = 1$.

(i) [3 points] Show that this defines a partial order on Y.

Given an arbitrary chain C in Y, define $C' = C \times \{1\} \in V^{(B)}$.

(ii) [2 points] Show that the formula expressing that C' is a chain in X is true in $V^{(B)}$.